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Optimal design and control of composite laminated plates with various boundary conditions using various plate theories

M.E. Fares *, Y.G. Youssif, A.E. Alamir

Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt

Abstract

The optimal control problem of minimizing the dynamic response of anisotropic symmetric or antisymmetric composite laminated rectangular plates with various boundary conditions is presented using various plate theories. The objective of the present control problem is to minimize the dynamic response of the plate with minimum possible expenditure of force. The dynamic response of the structure comprises a weight sum of the control objective (the total vibrational energy) and a penalty functional of the control force. In addition to the active control, the layer thickness and the orientation angle of the material fibers are taken as optimization design variables. The explicit solutions for the optimal force and controlled deflections are obtained in forms of double series using the Liapunov–Bellman theory. The effectiveness of the proposed control and the behavior of the controlled structure are investigated. Various numerical results including the effect of boundary conditions, number of layers, anisotropy ratio, aspect ratio, and side-to-thickness ratio on the control process for symmetric and antisymmetric laminates are presented.

Keywords: Minimizing the dynamic response; Control force; Optimal design; Composite laminated plates; Various plate theories

1. Introduction

Laminated composite structures are made up of two or more layers bonded together to achieve the best properties of the constituent layers. By altering the material or the lamination scheme of the layers, a structural designer can tailor the strength and other suitable properties of layered structures to serve useful functions under certain conditions. Therefore, during the past two decades, a great deal of interest has been devoted to integrating the optimal design and active control in a single formulation. This topic is the subject of several research studies [1,2].

The studies [3–6] on the design of vibrating structures and on the active control of dynamically loaded structures treat the design optimization and structural control problems as separate issues. The integrated approach to the problem in which design and control are optimized simultaneously has been employed in several research studies [7–16]. In the works [7–10], the design and control problem was formulated as a constrained optimization problem. A multiobjective optimization approach was used in [11–15] with constraints imposed on relevant quantities. In the work [16], structural optimization was employed taking the fiber orientation, and stiffener areas as design variables to improve a control performance index using a sequential optimization procedure. More recent studies may be found in the literature [17–29]. For these works, the control formulations for composite laminated plates are presented based on the classical laminate theories for special boundary conditions and few papers have been formulated based on higher-order laminate theories with various cases of boundary conditions.

The current work deals with the optimal design and control of the dynamic response of an anisotropic rectangular composite laminate with various cases of boundary conditions. The present control formulation is based on a consistent higher-order plate theory [30]. The objective of the present control problem is to minimize the dynamic response (vibrational total energy) with minimum expenditure of force. Furthermore, the orientation angle of the material fiber and the layer thickness are taken as design variables. Control over the plate is exerted by distributed forces, which translate into force in the actual implementation of the control mechanism. The dynamic response is related to the energy of the structure, which is subject to initial

* Corresponding author. Tel.: +20-50-346-781; fax: +20-50-346-254.
E-mail address: sinfac@mum.mans.eun.eg (M.E. Fares).

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disturbances. A quadratic functional of the total energy is specified as the control performance index. The expenditure of force is limited by attaching a functional of the force to the objective functional as a penalty term. The necessary and sufficient conditions for optimal stabilization in the Liapunov–Bellman sense [31,32] are used to determine the optimal control force and deflections. Numerical examples are given to assess the present design–control approach for composite laminated plates with various boundary conditions.

2. Formulation of the problem

Consider a fiber reinforced rectangular laminated plate of length a, width b, and total thickness h, and composed of N anisotropic homogeneous layers bonded together in an arbitrary lamination scheme. The material of each layer is assumed to possess one plane of elastic symmetry parallel to the mid-plane of the plate. The coordinate system is taken such that the mid-plane coincides with the xy plane and is normal to the z-axis. Let the upper surface of the plate ($z = h/2$) be subjected to a transverse distributed load $q(x, y, t)$ which may be taken as a control force. Also, the initial conditions are specified as

$$w(x,y,0) = \overline{A}(x,y), \quad \dot{w}(x,y,0) = \overline{B}(x,y). \tag{1}$$

The superposed dot denotes the differentiation with respect to time t. The present study accounts for a displacement field which preserves the transverse shear stresses vanishing on the plate top and bottom surfaces [30] and is given by

$$u_1 = u + z\left[\alpha\frac{\partial w}{\partial x} + \beta\psi + \gamma z^2\left(\frac{\partial w}{\partial x} + \psi\right)\right], \tag{2a}$$

$$u_2 = v + z\left[\alpha\frac{\partial w}{\partial y} + \beta\phi + \gamma z^2\left(\frac{\partial w}{\partial y} + \phi\right)\right], \tag{2b}$$

$$u_3 = w, \tag{2c}$$

where (u_1, u_2, u_3) are the displacements along x, y and z directions, respectively, (u, v, w) are the displacements of a point on the mid-plane, and ψ and ϕ are the slopes in the xz and yz planes due to bending only (slope rotations).

The above displacement field (2a)–(2c) is the most general consistent higher-order displacement field which gives all other theories. The following lower-order theories can be obtained as:

1. higher-order plate theory (*HPT*): $\alpha = 0$, $\beta = 1$, $\gamma = -4/(3h^2)$;
2. first-order plate theory (*FPT*): $\alpha = 0$, $\beta = 1$, $\gamma = 0$;
3. classical plate theory (*CPT*): $\alpha = -1$, $\beta = 0$, $\gamma = 0$.

Applying Hamilton's principle, the governing equations of the laminate can be given in the form [33]:

$$\frac{\partial N_1}{\partial x} + \frac{\partial N_6}{\partial y} = I_1\ddot{u} + \hat{I}_2\ddot{\psi} + \overline{I}_2\frac{\partial\ddot{w}}{\partial x}, \tag{3a}$$

$$\frac{\partial N_6}{\partial x} + \frac{\partial N_2}{\partial y} = I_1\ddot{v} + \hat{I}_2\ddot{\phi} + \overline{I}_2\frac{\partial\ddot{w}}{\partial y}, \tag{3b}$$

$$\begin{aligned}\frac{\partial\overline{Q}_5}{\partial x} + \frac{\partial\overline{Q}_4}{\partial y} + q - \frac{\partial^2\overline{M}_1}{\partial x^2} - 2\frac{\partial^2\overline{M}_6}{\partial x\partial y} - \frac{\partial^2\overline{M}_2}{\partial y^2} \\ = I_1\ddot{w} - \overline{I}_2\left(\frac{\partial\ddot{u}}{\partial x} + \frac{\partial\ddot{v}}{\partial y}\right) - (\alpha\overline{I}_3 + \gamma\overline{I}_5)\left(\frac{\partial^2\ddot{w}}{\partial x^2} + \frac{\partial^2\ddot{w}}{\partial y^2}\right) \\ - (\beta\overline{I}_3 + \gamma\overline{I}_5)\left(\frac{\partial\ddot{\psi}}{\partial x} + \frac{\partial\ddot{\phi}}{\partial y}\right),\end{aligned} \tag{3c}$$

$$\begin{aligned}\frac{\partial\hat{M}_1}{\partial x} + \frac{\partial\hat{M}_6}{\partial y} = \hat{Q}_5 + \hat{I}_2\ddot{u} + \left(\beta\hat{I}_3 + \gamma\hat{I}_5\right)\ddot{\psi} \\ + \left(\alpha\hat{I}_3 + \gamma\hat{I}_5\right)\frac{\partial\ddot{w}}{\partial x},\end{aligned} \tag{3d}$$

$$\begin{aligned}\frac{\partial\hat{M}_6}{\partial x} + \frac{\partial\hat{M}_2}{\partial y} = \hat{Q}_4 + \hat{I}_2\ddot{v} + \left(\beta\hat{I}_3 + \gamma\hat{I}_5\right)\ddot{\phi} \\ + \left(\alpha\hat{I}_3 + \gamma\hat{I}_5\right)\frac{\partial\ddot{w}}{\partial y},\end{aligned} \tag{3e}$$

the stress resultants N_i and M_i, etc. can be expanded as:

$$(N_i, M_i, P_i) = \sum_{k=1}^{N}\int_{z_{k-1}}^{z_k}(1, z, z^3)\sigma_i^k\,\mathrm{d}z \quad (i = 1, 2, 6), \tag{4a}$$

$$(Q_i, R_i) = \sum_{k=1}^{N}\int_{z_{k-1}}^{z_k}(1, z^2)\sigma_i^k\,\mathrm{d}z \quad (i = 4, 5), \tag{4b}$$

$$\overline{Q}_j = (1+\alpha)Q_j + 3\gamma R_j, \quad \hat{Q}_j = \beta Q_j + 3\gamma R_j,$$
$$\overline{M}_j = \alpha M_j + \gamma P_j, \quad \hat{M}_j = \beta M_j + \gamma P_j,$$
$$\overline{I}_n = \alpha I_n + \gamma I_{n+2}, \quad \hat{I}_n = \beta I_n + \gamma I_{n+2},$$
$$I_n = \sum_{k=1}^{N}\int_{z_{k-1}}^{z_k}\rho^{(k)}z^{n-1}\,\mathrm{d}z,$$

where z_k and z_{k-1} are the top and bottom z-coordinates of the kth layer, σ_i $(i = 1, 2, \ldots, 6)$ denote the stress components, $\sigma_1 = \sigma_{11}$, $\sigma_2 = \sigma_{22}$, $\sigma_3 = \sigma_{33}$, $\sigma_4 = \sigma_{23}$, $\sigma_5 = \sigma_{13}$, $\sigma_6 = \sigma_{12}$. The stress resultants are related to the strain components by the following laminate constitutive equations:

$$\begin{aligned}N_i &= A_{ij}\varepsilon_j^{(0)} + B_{ij}\varepsilon_j^{(1)} + E_{ij}\varepsilon_j^{(3)}, \\ M_i &= B_{ij}\varepsilon_j^{(0)} + D_{ij}\varepsilon_j^{(1)} + F_{ij}\varepsilon_j^{(3)}, \\ P_i &= E_{ij}\varepsilon_j^{(0)} + F_{ij}\varepsilon_j^{(1)} + H_{ij}\varepsilon_j^{(3)} \\ &\quad (i, j = 1, 2, 6),\end{aligned} \tag{5a}$$

$$\begin{aligned}Q_i &= A_{ij}\varepsilon_j^{(0)} + D_{ij}\varepsilon_j^{(2)}, \\ R_i &= D_{ij}\varepsilon_j^{(0)} + F_{ij}\varepsilon_j^{(2)}, \\ &\quad (i, j = 4, 5).\end{aligned} \tag{5b}$$

The homogeneous laminate stiffnesses $A_{ij}, B_{ij}, \ldots$ are in the form:

$$(A_{ij},B_{ij},D_{ij},E_{ij},F_{ij},H_{ij}) = \sum_{k=1}^{N}\int_{z_{k-1}}^{z_k} C_{ij}^{(k)}(1,z,z^2,z^3,z^4,z^6)\mathrm{d}z \quad (i,j=1,2,6), \tag{6a}$$

$$(A_{ij},D_{ij},F_{ij}) = \sum_{k=1}^{N}\int_{z_{k-1}}^{z_k} C_{ij}^{(k)}(1,z^2,z^4)\mathrm{d}z \quad (i,j=4,5), \tag{6b}$$

where $C_{ij}^{(k)}$ are the material stiffnesses of the kth layer which depend on material properties and orientation angle θ_k of the layer material, and ε_j are the infinitesimal strains associated with the displacements (2a)–(2c) given by:

$$\begin{aligned}
&\varepsilon_i = \varepsilon_i^{(0)} + z\varepsilon_i^{(1)} + z^3\varepsilon_i^{(3)}, \quad \varepsilon_3 = 0, \quad \varepsilon_j = \varepsilon_j^{(0)} + z^2\varepsilon_j^{(2)} \quad (i=1,2,6;\ j=4,5),\\
&\varepsilon_1^{(0)} = \frac{\partial u}{\partial x}, \quad \varepsilon_2^{(0)} = \frac{\partial v}{\partial y}, \quad \varepsilon_4^{(0)} = (1+\alpha)\frac{\partial w}{\partial y} + \beta\phi,\\
&\varepsilon_5^{(0)} = (1+\alpha)\frac{\partial w}{\partial x} + \beta\psi, \quad \varepsilon_6^{(0)} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y},\\
&\varepsilon_1^{(1)} = \alpha\frac{\partial^2 w}{\partial x^2} + \beta\frac{\partial \psi}{\partial x}, \quad \varepsilon_2^{(1)} = \alpha\frac{\partial^2 w}{\partial y^2} + \beta\frac{\partial \phi}{\partial y},\\
&\varepsilon_6^{(1)} = 2\alpha\frac{\partial^2 w}{\partial x\partial y} + \beta\left(\frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial y}\right), \quad \varepsilon_1^{(3)} = \frac{1}{3}\frac{\partial \varepsilon_4^{(2)}}{\partial x},\\
&\varepsilon_4^{(2)} = 3\gamma\left(\frac{\partial w}{\partial y} + \phi\right), \quad \varepsilon_2^{(3)} = \frac{1}{3}\frac{\partial \varepsilon_5^{(2)}}{\partial y},\\
&\varepsilon_6^{(3)} = \gamma\left(2\frac{\partial^2 w}{\partial x\partial y} + \frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial y}\right), \quad \varepsilon_5^{(2)} = 3\gamma\left(\frac{\partial w}{\partial x} + \psi\right).
\end{aligned} \tag{7}$$

The present control problem accounts for various cases of boundary conditions at the edges, i.e., when the plate edges are simply supported (S), or clamped (C) or free (F), or when a combination of these boundary conditions is prescribed over the edges. Then, these boundary conditions on the edges perpendicular to x-axis take the form:

$$\begin{aligned}
&S: \ v = w = \phi = N_1 = \hat{M}_1 = P_1 = 0,\\
&C: \ u = v = w = \psi = \phi = w_{,x} = 0,\\
&F: \ N_1 = \hat{M}_1 = P_1 = N_6 = M_6 - P_6 = \hat{Q}_1 + P_{1,x} + P_{6,y} = 0,
\end{aligned} \tag{8}$$

where $(\ )_{,x}$ denotes the partial differentiation with respect to x.

3. The optimal control problem

The objective of the present study is to minimize the dynamic response of the laminate in a specified time $0 \leqslant t \leqslant \tau \leqslant \infty$ with the minimum possible expenditure of force $q(x,y,t)$. The dynamic response of the plate is measured by a cost functional related to the energy of the system which is a function of displacements, its spatial derivatives and the velocity. The optimization variable $q(x,y,t)$ may be introduced in the objective functional by taking a performance index which compress a weight sum of plate energy and a penalty functional of the control force. In addition to the active control using the force $q(x,y,t)$, we consider the layer thickness h_k and the orientation angle θ_k as optimization design variables. Then, the mathematical formulation of the control problem can be reduced to determine the optimal control variables q, h_k, and θ_k that minimize the functional

$$J = \mu_1 J_1 + \mu_2 J_2 + \mu_3 J_3, \tag{9}$$

$$J_1(q) = \frac{1}{2}\int_0^\infty\int_0^a\int_0^b\int_{-h/2}^{h/2}(\varepsilon_1\sigma_1 + \varepsilon_2\sigma_2 + \varepsilon_4\sigma_4 + \varepsilon_5\sigma_5 + \varepsilon_6\sigma_6)\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t, \tag{10a}$$

$$J_2(q) = \frac{1}{2}\int_0^\infty\int_0^a\int_0^b\int_{-h/2}^{h/2}\rho^{(k)}\left(\dot{u}_1^2 + \dot{u}_2^2 + \dot{u}_3^2\right)\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t, \tag{10b}$$

$$J_3(q) = \int_0^\tau\int_0^b\int_0^a q^2(x,y,t)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t, \tag{10c}$$

where $\mu_i > 0$, $i = 1,2,3$, are the constant weighting factors, J_1 and J_2 represent the strain energy and the kinetic energy of the laminate, respectively. The functional J_3 is a penalty term involving the control function $q \in L^2$, where L^2 denotes the set of all bounded square integrable functions on $\{0 \leqslant x \leqslant a, 0 \leqslant y \leqslant b, 0 \leqslant t \leqslant \tau \leqslant \infty\}$.

4. Solution procedure

The solution of the system of partial differential equations (3a)–(3e) under conditions (1) and (8) may be expanded in the form of double series in terms of the free vibration eigenfunctions. Then the displacement functions (u,v,w,ψ,ϕ) and the closed-loop control function q may be represented as

$$\begin{aligned}
&u = \sum_{m,n} U_{mn}(t)XY_{,y}, \quad v = \sum_{m,n} V_{mn}(t)X_{,x}Y,\\
&w = \sum_{m,n} W_{mn}(t)XY, \quad \psi = \sum_{m,n} \Psi_{mn}(t)X_{,x}Y,\\
&\phi = \sum_{m,n} \Phi_{mn}(t)XY_{,y}, \quad q = \sum_{m,n} Q_{mn}(t)XY,
\end{aligned} \tag{11}$$

where U_{mn}, V_{mn}, W_{mn}, Ψ_{mn}, Φ_{mn} and Q_{mn} are unknown functions of time. The functions $X(x)$ and $Y(y)$ are continuous orthonormed functions, which satisfy at least the geometric boundary conditions given in (8), and represent approximate shapes of the deflected surface of the vibrating plate. These functions, for the different cases of boundary conditions, take the following forms [34]:

$$
\begin{aligned}
SS:\ & X(x) = \sin\mu_m x, \quad \mu_m = m\pi/a.\\
CC:\ & X(x) = \sin\mu_m x - \sinh\mu_m x\\
& \qquad - \eta_m(\cos\mu_m x - \cosh\mu_m x),\\
& \eta_m = (\sin\mu_m a - \sinh\mu_m a)\\
& \qquad \times (\cos\mu_m a - \cosh\mu_m a)^{-1},\\
& \mu_m = (m+0.5)\pi/a.\\
CS:\ & X(x) = \sin\mu_m x - \sinh\mu_m x\\
& \qquad - \eta_m(\cos\mu_m x - \cosh\mu_m x),\\
& \eta_m = (\sin\mu_m a + \sinh\mu_m a)\\
& \qquad \times (\cos\mu_m a + \cosh\mu_m a)^{-1},\\
& \mu_m = (m+0.25)\pi/a.\\
CF:\ & X(x) = \sin\mu_m x - \sinh\mu_m x\\
& \qquad - \eta_m(\cos\mu_m x - \cosh\mu_m x),\\
& \eta_m = (\sin\mu_m a + \sinh\mu_m a)\\
& \qquad \times (\cos\mu_m a + \cosh\mu_m a)^{-1},\\
& \mu_1 = 1.875/a, \quad \mu_2 = 4.694/a,\\
& \mu_3 = 7.855/a,\\
& \mu_4 = 10.996/a \quad \text{and}\\
& \mu_m = (m-0.25)\pi/a \ \text{ for } m \geqslant 5.
\end{aligned}
\tag{12}
$$

Using Eqs. (5a), (5b) and (7), we can get the governing equations (3a)–(3e) in terms of the displacements. For these equations, the in-plane inertia terms may be neglected. Substituting expressions (11) into the resulting equations and multiplying each equation by the corresponding eigenfunction, then integrating over the domain of solution, we obtain after some mathematical manipulations, the following time equations:

$$
\begin{aligned}
& U_{1mn}U_{mn} + V_{1mn}V_{mn} + W_{1mn}W_{mn} + \Psi_{1mn}\Psi_{mn}\\
& \quad + \Phi_{1mn}\Phi_{mn} = 0,\\
& U_{2mn}U_{mn} + V_{2mn}V_{mn} + W_{2mn}W_{mn} + \Psi_{2mn}\Psi_{mn}\\
& \quad + \Phi_{2mn}\Phi_{mn} = 0,\\
& U_{3mn}U_{mn} + V_{3mn}V_{mn} + W_{3mn}W_{mn} + \Psi_{3mn}\Psi_{mn}\\
& \quad + \Phi_{3mn}\Phi_{mn} = \overline{W}_{1mn}\ddot{W}_{mn} - Q_{mn},\\
& U_{4mn}U_{mn} + V_{4mn}V_{mn} + W_{4mn}W_{mn} + \Psi_{4mn}\Psi_{mn}\\
& \quad + \Phi_{4mn}\Phi_{mn} = \overline{W}_{2mn}\ddot{W}_{mn},\\
& U_{5mn}U_{mn} + V_{5mn}V_{mn} + W_{5mn}W_{mn} + \Psi_{5mn}\Psi_{mn}\\
& \quad + \Phi_{5mn}\Phi_{mn} = \overline{W}_{3mn}\ddot{W}_{mn},
\end{aligned}
\tag{13}
$$

where

$$
\begin{aligned}
& \overline{W}_{1mn} = I_1 e_7 - (e_{13} + e_{16})\left(\alpha\bar{I}_3 + \gamma\bar{I}_5\right),\\
& \overline{W}_{2mn} = e_6\left(\alpha\hat{I}_3 + \gamma\hat{I}_5\right), \quad \overline{W}_{3mn} = e_{12}\left(\alpha\hat{I}_3 + \gamma\hat{I}_5\right),
\end{aligned}
$$

the coefficients U_{imn}, V_{imn}, W_{imn}, Φ_{imn} and Ψ_{imn} $(i = 1, 2, 3)$ are given in Appendix A. Solving the system (13), one gets an equation of the time-dependent functions W_{mn} and Q_{mn} only,

$$
\begin{aligned}
& \ddot{W}_{mn} + \omega_{mn}^2 W_{mn} = l_{m,n} Q_{mn},\\
& \omega_{mn}^2 = \frac{\Delta_{mn}}{\Delta_{1mn}}, \quad l_{mn} = \frac{\Delta_0}{\Delta_{1mn}},
\end{aligned}
\tag{14}
$$

where Δ_{mn}, Δ_{1mn} and Δ_0 are given in Appendix A.

Following previous analogous steps, we can get the objective functional (9) in the final form

$$
\begin{aligned}
J = \sum_{m,n} \int_0^\infty & \Big(k_1 W_{mn}^2 + k_2 W_{mn} Q_{mn} + k_3 Q_{mn}^2 + k_4 \dot{W}_{mn}^2\\
& + k_5 \dot{W}_{mn}\dot{Q}_{mn} + k_6 \dot{Q}_{mn}^2\Big)\,\mathrm{d}t,
\end{aligned}
\tag{15}
$$

where the coefficients k_i $(i = 1, 2, \ldots, 6)$ are given in Appendix A. Since the system (14) is separable, hence the functional (15) depends only on the variables found in (m, n)th equation of the system. With the aid of this condition, the problem is reduced to a problem of analytical design of controllers [35] for every $m, n = 1, 2, \ldots, \infty$.

Now the optimal control problem is to find firstly the control function $q_{mn}^{\mathrm{opt}}(t)$ that satisfies the conditions

$$
J(q_{mn}^{\mathrm{opt}}) \leqslant J(q_{mn}) \quad \text{for all } q_{mn}(t) \in L^2([0, \infty]),
$$

that is

$$
\min_{q_{mn}} J = \min \sum J_{mn} = \sum_{m,n} \min_{q_{mn}\in L^2} J,
$$

and secondly, to find the optimal values of h_k and θ_k from the following minimization condition:

$$
\begin{aligned}
& J(q_{mn}^{\mathrm{opt}}, h_k^{\mathrm{opt}}, \theta_k^{\mathrm{opt}}) = \min_{h_k,\theta_k} I(q_{mn}^{\mathrm{opt}}, h_k, \theta_k),\\
& \sum_k h_k = h, \quad 0 < \theta < \pi/2.
\end{aligned}
$$

Here, the minimization of the dynamic response using the control force q_{mn} can be carried out independently for every modal equation. For such a problem, Liapunov–Bellman theory [32] is considered an effective approach for the solution. The necessary and sufficient conditions for minimizing the functional (15) according to the Liapunov–Bellman theory is

$$
\min_q \left[\frac{\partial L_{mn}}{\partial W_{mn}}\dot{W}_{mn} + \frac{\partial L_{mn}}{\partial \dot{W}_{mn}}\ddot{W}_{mn} + \overline{J}_{mn}\right] = 0,
\tag{16}
$$

provided that the Liapunov function L_{mn}

$$
L_{mn} = A_{mn} W_{mn}^2 + 2B_{mn} W_{mn}\dot{W}_{mn} + C_{mn}\dot{W}_{mn}^2,
\tag{17}
$$

is a positive definite, i.e. $A_{mn} > 0$, $C_{mn} > 0$ and $A_{mn}C_{mn} > B_{mn}^2$, where $\overline{J}_{mn}$ is the integrand of (15). Using Eq. (17) we can obtain the optimal control function in the form:

$$
Q_{mn}^{\mathrm{opt}} = \frac{-1}{2k_3}(2B_{mn} l_{mn} + k_2) W_{mn} - \frac{C_{mn} l_{mn}}{k_3}\dot{W}_{mn}.
\tag{18}
$$

Then, substituting Eq. (18) into (16) and equating the coefficients of W_{mn}^2, $\dot{W}_{mn}^2$ and $W_{mn}\dot{W}_{mn}$ by zeroes, the following system of equations is obtained:

$$\begin{aligned} &C_{mn}^2(a_1B_{mn}^2 + a_2B_{mn} + a_3) + a_4B_{mn} + a_5 = 0,\\ &C_{mn}^2(a_6C_{mn}^2 + a_7B_{mn} + a_8) + a_9B_{mn}^2\\ &\quad + a_{10}B_{mn} + a_{11} = 0,\\ &a_{12}A_{mn} + C_{mn}(a_{13}C_{mn}^2 + a_{14}C_{mn}^2B_{mn}\\ &\quad + a_{15}B_{mn}^2 + a_{16}B_{mn} + a_{17}) = 0. \end{aligned} \tag{19}$$

Under the condition that the Liapunov function is a positive definite, the solution of the system of nonlinear algebraic equations (19) may be obtained. Then, when $\overline{B}(x,y) = 0$, we can get the controlled deflection solution in the form:

$$\begin{aligned} &W_{mn} = A^* e^{l_{mn}C_{2mn}t/2}\left(\cos(v_{mn}t) - \frac{C_{2mn}}{v_{mn}}\sin(v_{mn}t)\right),\\ &v_{mn} = \left(\omega_{mn}^2 - l_{mn}C_{1mn} - \frac{1}{4}l_{mn}^2C_{2mn}^2\right)^{1/2},\\ &C_{1mn} = -\frac{1}{2k_3}(2B_{mn}l_{mn} + k_2), \quad C_{2mn} = -\frac{1}{k_3}C_{mn}l_{mn}, \end{aligned} \tag{20}$$

where A^* is the amplitude of the initial deflection. Inserting these expressions into (13), (15) and (18) we can get the displacements, the total energy and the optimal control force. Then, we complete the minimization process for the dynamic response of the laminate by determining the optimal design of the laminate using the design variables θ_k and h_k.

5. Numerical results and discussion

Numerical results for maximum optimal control force q, controlled deflection W and total energy J are presented for symmetric and antisymmetric angle-ply rectangular plates with various cases of the boundary conditions (8). All layers of the laminate are assumed to be of the same orthotropic materials. A shear correction factor for *FPT* is taken to be 5/6. The plane reduced stress material stiffnesses C_{ij} are given by:

$$\begin{aligned} &C_{11} = \frac{E_1}{1 - v_{12}v_{21}}, \quad C_{12} = \frac{v_{12}E_2}{1 - v_{12}v_{21}},\\ &C_{22} = \frac{E_2}{1 - v_{12}v_{21}}, \quad C_{44} = G_{23}, \quad C_{55} = G_{13},\\ &C_{66} = G_{12}, \quad v_{ij}E_j = v_{ji}E_i \qquad (i,j = 1,2), \end{aligned}$$

where E_i are Young's moduli, v_{ij} are Poisson's ratios and G_{ij} are the shear moduli. In all calculations, unless otherwise stated, the following parameters are used:

$$\begin{aligned} &a = b = 20 \text{ in.}, \quad h = 2 \text{ in.}, \quad \rho = 0.00012 \text{ Ib s}^2/\text{in.}^4,\\ &\mu_1 = \mu_2 = 1, \quad \mu_3 = 0.001, \quad \overline{A} = 10^3 l\omega^{-2},\\ &E_2 = 10^6 \text{ psi}, \quad E_1 = 25E_2, \quad G_{12} = G_{13} = 0.5E_2,\\ &G_{23} = 0.2E_2, \quad v_{12} = 0.25. \end{aligned}$$

For the optimal design, we consider a $(\theta, 0, \theta)$ plate with outer layers having the same thickness; and therefore we take the optimization thickness variable r which represents the ratio of the outer layer thickness to the total laminate thickness. All calculations in tables and figures are carried out at $x = a/2$, $y = b/2$ and for a maximum amplitude of W and q.

Table 1 contains values of the maximum optimal control force q, controlled deflection W and total energy J for simply supported (*SSSS*) three-, five- and thirteen-layer symmetric square plates. These values are obtained using classical, first-order and higher-order plate theories (*CPT*, *FPT*, *HPT*) for some values of the side-to-thickness ratio a/h. Table 2 presents similar results for two-, four- and twelve-layer antisymmetric square plates. Note that both *FPT* and *HPT* give almost the same values with a slight deviation occurring for moderately thick plates ($a/h < 10$). The *CPT* underpredicts q, J and W with minimum errors reaching 6% for thin plates and more 40% for thick plates ($a/h \leqslant 5$).

Tables 3 and 4 contain values of q, W and J for multilayered symmetric and antisymmetric laminates for

Table 1
q, J and W for three-, five- and thirteen-layer symmetric *SSSS* plates according to *CPT*, *FPT* and *HPT*, $a = b = 20$, $E_1/E_2 = 25$

a/h	*Th.*	$45, 0, 45$			$45, -45, 0, -45, 45$			$45, -45, 45, -45, 45, -45, /0/_{\text{sym}}$		
		q	J	W	q	J	W	q	J	W
5	*CPT*	58.057	.54762	.00600	57.692	.54040	.0059	57.598	.53856	.00590
	FPT	103.64	1.8326	.02068	103.19	1.8152	.0204	102.97	1.8067	.02039
	HPT	109.83	2.0882	.02353	108.76	2.0438	.0230	108.10	2.0170	.02274
10	*CPT*	150.86	3.0423	.04806	150.00	3.0020	.0474	149.77	2.9917	.04727
	FPT	185.28	4.8899	.07746	184.38	4.8330	.0765	184.06	4.8125	.07625
	HPT	191.64	5.3148	.08410	190.21	5.2193	.0826	189.47	5.1700	.08183
20	*CPT*	341.55	17.849	.38454	340.01	17.603	.3794	339.62	17.540	.37818
	FPT	358.38	20.698	.44336	356.88	20.423	.4377	356.44	20.343	.43614
	HPT	361.94	21.378	.45712	360.16	21.042	.4503	359.49	20.917	.44775

Table 2
q, J and W for two, four and twelve-layer antisymmetric *SSSS* plates according to *CPT*, *FPT* and *HPT*, $a = b = 20$, $E_1/E_2 = 25$

a/h	*Th.*	$45, -45$			$45, -45, 45, -45$			$45, -45, 45, -45, 45, -45, /_{antisym}$		
		q	J	W	q	J	W	q	J	W
5	*CPT*	90.785	1.4060	.01526	62.474	.63748	.00697	58.080	.54792	.00601
	FPT	122.76	2.6522	.02973	105.42	1.9015	.02145	103.17	1.8146	.02048
	HPT	122.98	2.6788	.03001	112.04	2.1821	.02458	108.46	2.0316	.02290
10	*CPT*	223.48	7.8240	.12211	161.04	3.5386	.05581	150.89	3.0434	.04808
	FPT	243.48	9.6541	.15106	192.46	5.3583	.08476	184.85	4.8623	.07703
	HPT	244.02	9.7332	.15221	198.76	5.8075	.09175	190.33	5.2271	.08272
20	*CPT*	455.25	48.472	.97692	359.06	20.883	.44653	341.58	17.854	.38465
	FPT	462.73	51.590	1.0348	373.78	23.722	.50442	358.17	20.658	.44255
	HPT	462.99	51.743	1.0374	377.10	24.442	.51879	361.23	21.243	.45437

Table 3
Effect of the number of layers N on q, J and W for symmetric $(45, -45, \ldots, 0, \ldots, -45, 45)$ plates according to *CPT*, *FPT* and *HPT* with various boundary conditions, $a = b = 20$, $a/h = 5$, $E_1/E_2 = 25$

N	*Th.*	*CCSS*			*CCCC*			*CFSS*		
		q	J	W	q	J	W	q	J	W
3	*CPT*	76.329	.44143	.00543	120.32	.48222	.00616	301.55	24.068	.06026
	FPT	183.31	2.3881	.02617	339.80	3.4652	.03684	345.01	38.879	.10175
	HPT	191.07	2.6260	.02875	356.20	3.8642	.04102	358.37	44.536	.11511
5	*CPT*	77.305	.45023	.00550	120.76	.48438	.00617	302.75	24.180	.05997
	FPT	184.21	2.4141	.02645	339.62	3.4609	.03679	335.49	35.767	.09566
	HPT	191.85	2.6497	.02900	354.77	3.8294	.04066	344.07	39.328	.10515
13	*CPT*	77.563	.45257	.00552	120.88	.48495	.00617	303.06	24.209	.05989
	FPT	185.18	2.4422	.02675	339.42	3.4563	.03674	328.70	33.723	.09167
	HPT	193.23	2.6921	.02946	354.00	3.8106	.04046	332.11	35.458	.09753

Table 4
Effect of the number of layers N on q, J and W for antisymmetric $(45, -45, \ldots)$ plates according to *CPT*, *FPT* and *HPT* with various boundary conditions, $a = b = 20$, $a/h = 5$, $E_1/E_2 = 25$

N	*Th.*	*CCSS*			*CCCC*			*CFSS*		
		q	J	W	q	J	W	q	J	W
2	*CPT*	120.79	1.1418	.01383	187.83	1.2122	.01530	368.88	49.798	.12545
	FPT	211.23	3.2657	.03560	375.26	4.3304	.04586	408.09	71.931	.17477
	HPT	211.21	3.2830	.03577	374.64	4.3356	.04590	402.21	68.898	.17081
4	*CPT*	84.090	.53414	.00649	131.07	.57191	.00725	315.94	27.840	.06889
	FPT	189.11	2.5575	.02800	343.86	3.5585	.03781	340.05	37.569	.10059
	HPT	201.05	2.9411	.03215	365.08	4.0871	.04335	339.15	38.176	.10488
12	*CPT*	78.231	.46054	.00561	121.91	.49335	.00628	304.42	24.567	.06077
	FPT	186.21	2.4720	.02708	339.81	3.4653	.03683	326.50	33.126	.09065
	HPT	195.26	2.7554	.03014	355.32	3.8429	.04080	326.66	33.880	.09458

the *CCCC*, *CCSS* and *CFSS* boundary condition cases. These results extend the previous discussion on Tables 1 and 2 to various cases of boundary conditions. Moreover, all numerical results in Tables 1–4 indicate that the number of layers has a weak effect on q, J and W for the symmetric plates, and it more obvious for the antisymmetric ones. This is due to the fact that the generally orthotropic symmetric plates exhibit no coupling between bending and extension ($B_{ij} = 0$). In addition, the other coupling between normal forces and shearing strain, shearing force and normal strain; and normal moments and twist are not zeroes (i.e. $A_{16}, A_{26}, D_{16}, D_{26}, \ldots$ are not zeroes), but these stiffnesses decrease with increasing the number of layers; while, for the antisymmetric plate, the coupling stiffnesses B_{ij} are not all zeroes [36]. So, the bending–extension coupling makes the antisymmetric laminates more flexible and has deflection more than those of the symmetric ones

[34]. As a result, antisymmetric laminates need more expenditure of force to control their dynamic response.

Table 5 presents the optimal values of the orientation angle θ_{opt} and the thickness ratio r_{opt} for a three-layer symmetric square plate with various boundary conditions. It is evident that the optimal design strongly dependent on the aspect ratio and on the boundary conditions.

In general, the displayed numerical results in all figures are obtained using *HPT* for symmetric laminates $(\theta, 0, \theta)$ with various boundary conditions. Figs. 1–3 show the effect of the side-to-thickness ratio on the control force and total energy for *CSSS*, *CCSS* and *CFSS* boundary conditions. The q- and J-curves in these figures are plotted, in general for four cases of laminate designs which are nonoptimal design, optimal design using only the orientation angle θ, optimal design using only the thickness ratio r, and optimal design using both r and θ. Observe that all cases of optimal design considerably reduce the total energy of the laminate as compared to the uncontrolled ones, but the optimal design using θ only is more effective than that using r, and the optimal design using both r and θ is the most efficient. Moreover, these optimal designs are more

Table 5
The effect of aspect ratio *a*/*b* on the orientation angle θ_{opt}, and optimal thickness ratio r_{opt} of symmetric $(\theta, 0, \theta)$ laminates with various boundary conditions, according to *HPT*, $a/h = 10$, $E_1/E_2 = 25$

a/*b*	*SSSS*		*CSSS*		*CCSS*		*CCCC*		*CFSS*		*CFCF*	
	θ_{opt}	r_{opt}	θ_{opt}	r_{opt}	θ_{opt}	r_{opt}	θ_{opt}	r_{opt}	θ_{opt}	r_{opt}	θ_{opt}	r_{opt}
1	42.4°	0.5	33.1°	0.32	25.9°	0.23	35.3°	0.2	90°	0.5	0°	0
1.5	56.9°	0.5	52.6°	0.5	49.6°	0.47	62.9°	0.5	90°	0.5	90°	0.1
2	67.7°	0.5	64.8°	0.5	63.9°	0.5	80.5°	0.5	90°	0.5	90°	0.09
2.5	74.7°	0.5	72.5°	0.5	72.4°	0.5	90°	0.5	90°	0.5	90°	0.08
3	79.9°	0.5	78°	0.5	78.4°	0.5	90°	0.5	90°	0.5	90°	0.08
5	90°	0.5	90°	0.5	90°	0.5	90°	0.5	90°	0.5	90°	0.5

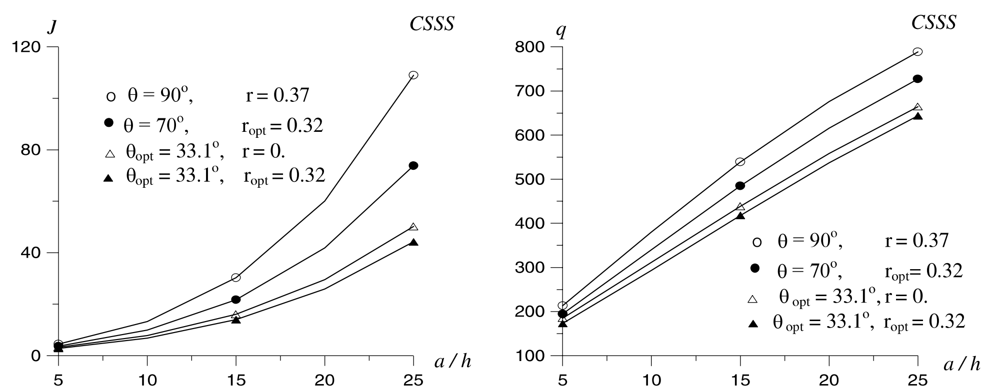

Fig. 1. Curves of J and q for $(\theta, 0, \theta)$ *CSSS* laminate, $a = b = 20, h = 2, E_1/E_2 = 25$.

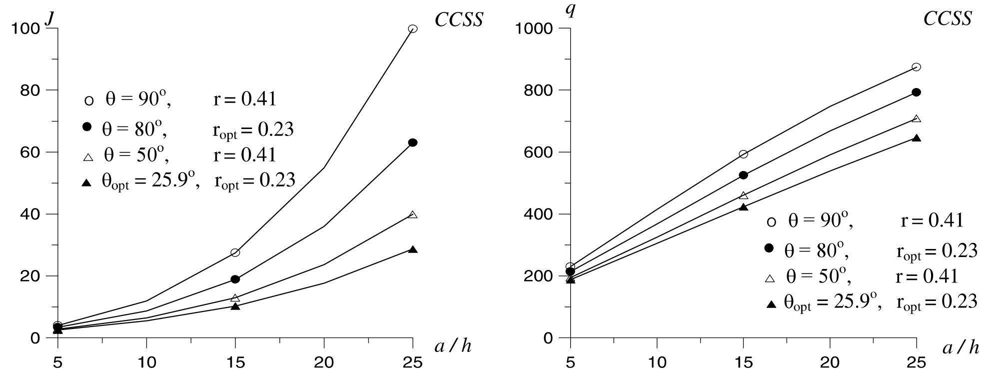

Fig. 2. Curves of J and q for $(\theta, 0, \theta)$ *CCSS* laminate, $a = b = 20, h = 2, E_1/E_2 = 25$.

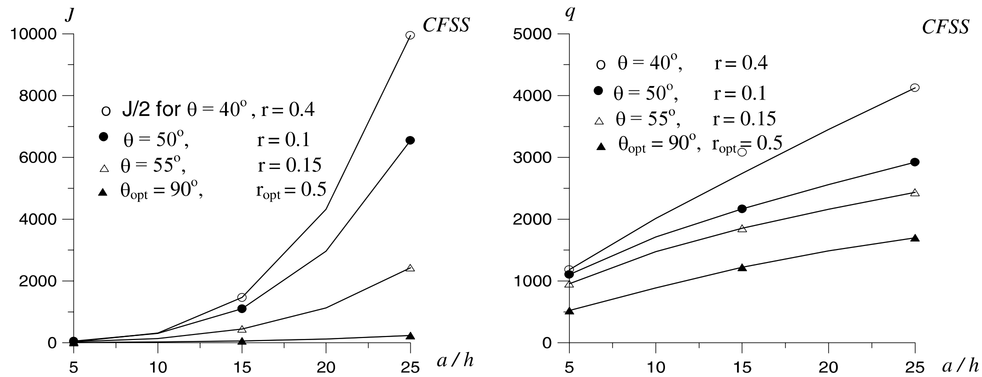

Fig. 3. Curves of J and q for $(\theta, 0, \theta)$ *CFSS* laminate, $a = b = 20, h = 2, E_1/E_2 = 25$.

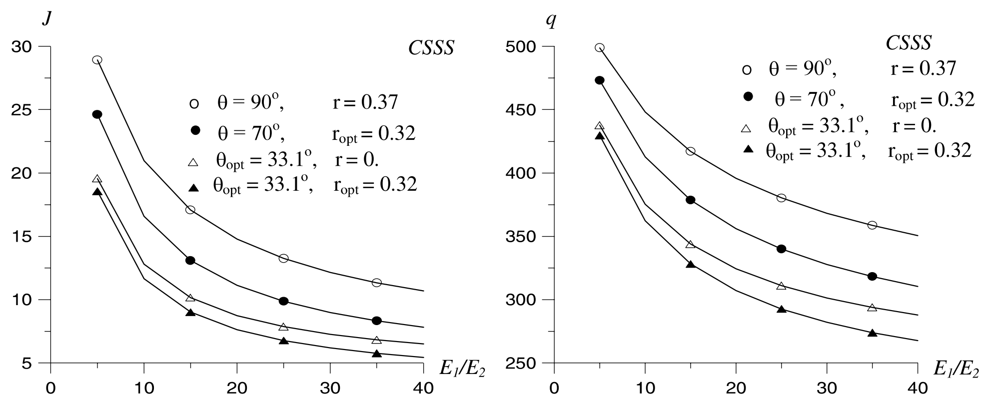

Fig. 4. Curves of J and q for $(\theta, 0, \theta)$ *CSSS*laminate, $a = b = 20, h = 2$.

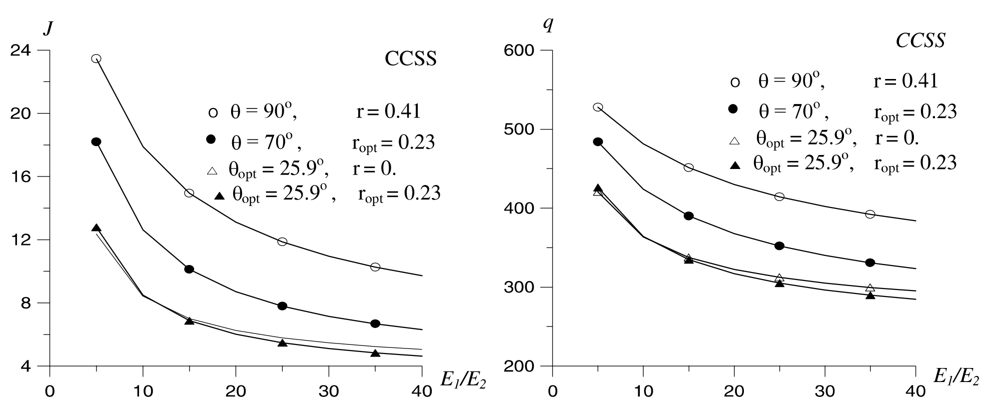

Fig. 5. Curves of J and q for $(\theta, 0, \theta)$ *CCSS* laminate, $a = b = 20, h = 2$.

significant and required for thin laminates $(a/h > 15)$, and for laminates, which have one or more free edges.

Figs. 4–6 show the effect of orthotropy ratio E_1/E_2 on J and q with various boundary conditions. These figures indicate that beside the role of the optimal design depending on r and θ for reducing the total energy J and the optimal control force q, the orthotropy ratio may play an important role in reducing them, where J and q are rapidly decreasing with increasing E_1/E_2. This is due to the fact that laminates with high orthotropy ratio E_1/E_2 are more stiff in bending and have smaller deflections when compared to the laminates of low orthotropy ratio. Fig. 7 contains q- and J-curves plotted versus the aspect ratio a/b for a three-layer symmetric *CFSS* plate; they show that the present optimal design and control approach is more desired for short plates $(a/b < 2)$.

6. Conclusions

Optimal design and control of composite laminates for minimizing the dynamic response with minimum possible expenditure of force are presented for various

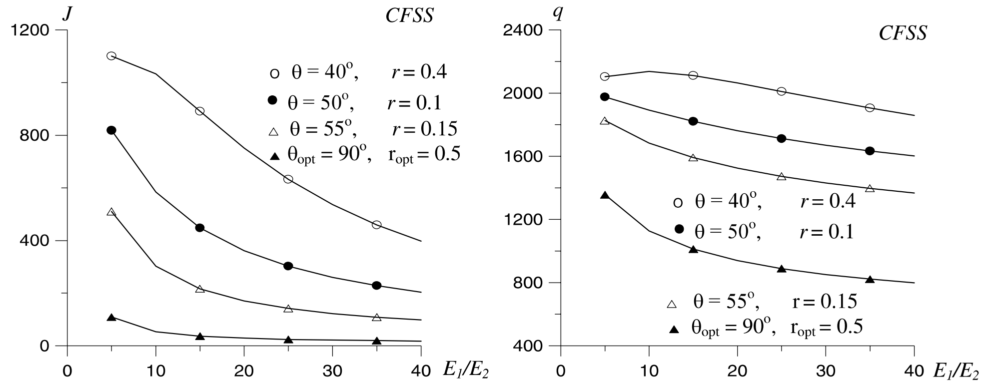

Fig. 6. Curves of J and q for $(\theta, 0, \theta)$ *CFSS* laminate, $a = b = 20, h = 2$.

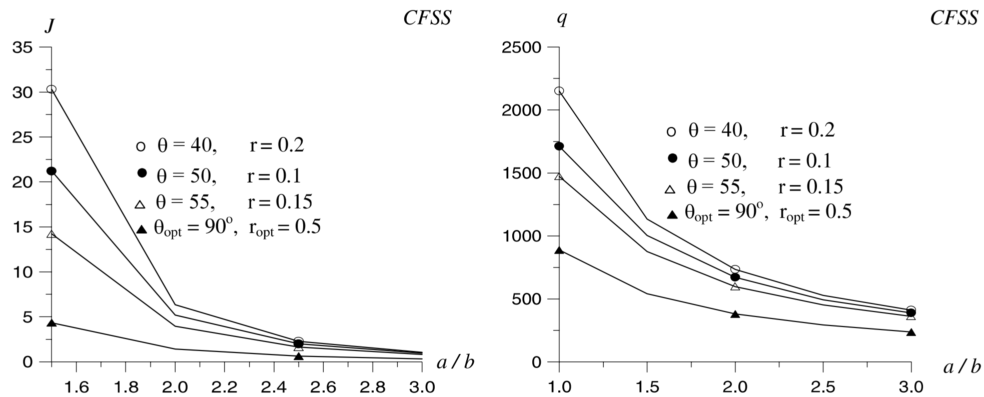

Fig. 7. Curves of J and q for $(\theta, 0, \theta)$ *CFSS* laminate, $a = 20, h = 2, E_1/E_2 = 25$.

cases of boundary conditions using various laminate theories. The study concludes that the classical theory underpredicts the control force and total energy with errors reaching more than 40% for moderately thick laminates, and the first-order laminate theory with a suitable shear correction factor gives results close to that obtained due to higher-order theory. The optimal design using the fiber orientation angle is more effective than that using the thickness of layers, and the optimal design using both fiber orientation and layer thickness is the most efficient. The present optimal design and control approach is more required for antisymmetric short laminates with low orthotropy ratio, in addition, the present control approach not only plays an efficient role in minimizing the dynamic response of the laminate, but also, it contributes significantly to decreasing the expenditure of control force.

Appendix A

$$(e_1, e_2, e_3, e_4, e_5, e_6) = \int_0^a \int_0^b \big(XY_{,yyy}, X_{,x}Y_{,yy}, X_{,xx}Y_{,y}, X_{,xxx}Y, XY_{,y}, X_{,x}Y\big) X_{,x}Y \, \mathrm{d}x \, \mathrm{d}y,$$

$$(e_8, e_9, e_{10}, e_{11}, e_{12}) = \int_0^a \int_0^b \big(XY_{,yyy}, X_{,xx}Y_{,y}, X_{,x}Y_{,yy}, X_{,xxx}Y, XY_{,y}\big) XY_{,y} \, \mathrm{d}x \, \mathrm{d}y,$$

$$(e_7, e_{13}, e_{14}, e_{15}, e_{16}, e_{17}) = \int_0^a \int_0^b \big(XY, X_{,xx}Y, X_{,xx}Y_{,yy}, X_{,xxxx}Y, XY_{,yy}, XY_{,yyyy}\big) XY \, \mathrm{d}x \, \mathrm{d}y,$$

$$(e_{18}, e_{19}, e_{20}) = \int_0^a \int_0^b \Big(X_{,xx}^2 Y^2, X_{,xx}Y_{,yy}XY, X^2 Y_{,yy}^2\Big) \mathrm{d}x \, \mathrm{d}y,$$

$$U_1 = A_{66}e_2 + 2A_{16}e_3 + A_{11}e_4,$$

$$\Psi_1 = \bar{s}_{66}e_2 + 2\bar{s}_{16}e_3 + \bar{s}_{11}e_4,$$

$$V_1 = A_{26}e_1 + (A_{66} + A_{12})e_2 + A_{16}e_3,$$

$$W_1 = s_{26}e_1 + (s_{12} + 2s_{66})e_2 + 3s_{16}e_3 + s_{11}e_4,$$

$$\Phi_1 = \bar{s}_{26}e_1 + (\bar{s}_{12} + \bar{s}_{66})e_2 + \bar{s}_{16}e_3,$$

$$U_2 = (A_{12} + A_{66})e_9 + A_{26}e_{10} + A_{16}e_{11},$$

$$V_2 = A_{22}e_8 + A_{66}e_9 + A_{26}e_{10},$$

$$\Phi_2 = \bar{s}_{22}e_8 + \bar{s}_{66}e_9 + 2\bar{s}_{26}e_{10},$$

$$W_2 = s_{22}e_8 + (s_{12} + 2s_{66})e_9 + 3s_{26}e_{10} + s_{16}e_{11},$$

$$\Psi_2 = (\bar{s}_{12} + \bar{s}_{66})e_9 + \bar{s}_{26}e_{10} + \bar{s}_{16}e_{11},$$

$$U_3 = -s_{26}e_1 - 3s_{16}e_{11} - (s_{12}+2s_{66})e_{14} - s_{11}e_{15},$$
$$V_3 = -3s_{26}e_1 - s_{16}e_{11} - (s_{12}+2s_{66})e_{14} + s_{22}e_{17},$$
$$W_3 = -4\eta_{26}e_1 + 2\zeta_{45}e_5 - 4\eta_{16}e_{11} + \zeta_{55}e_{13} - 2(\eta_{12}+2\eta_{66})e_{14} - \eta_{11}e_{15} + \zeta_{44}e_{16} - \eta_{22}e_{17},$$
$$\Psi_3 = -\overline{\eta}_{26}e_1 + \overline{\zeta}_{45}e_5 - \overline{\eta}_{16}e_{11} + \overline{\zeta}_{55}e_{13} - (\overline{\eta}_{12}+2\overline{\eta}_{66})e_{14} - \overline{\eta}_{16}e_{15},$$
$$\Phi_3 = -3\overline{\eta}_{26}e_1 + \overline{\zeta}_{45}e_5 - \overline{\eta}_{16}e_{11} + \overline{\zeta}_{44}e_{16} - (\overline{\eta}_{12}+2\overline{\eta}_{66})e_{14} - \overline{\eta}_{22}e_{17},$$

$$U_4 = \overline{s}_{66}e_2 + 2\overline{s}_{16}e_3 + \overline{s}_{11}e_4,$$
$$V_4 = \overline{s}_{26}e_1 + (\overline{s}_{12}+\overline{s}_{66})e_2 + \overline{s}_{16}e_3,$$
$$W_4 = \overline{\eta}_{26}e_1 + (\overline{\eta}_{12}+2\overline{\eta}_{66})e_2 + 3\overline{\eta}_{16}e_3 + \overline{\eta}_{11}e_4 - \overline{\zeta}_{45}e_5 - \overline{\zeta}_{55}e_6,$$
$$\Psi_4 = \eta^*_{66}e_2 + 2\eta^*_{16}e_3 + \eta^*_{11}e_4 - \zeta^*_{55}e_6,$$
$$\Phi_4 = \eta^*_{26}e_1 + \left(\eta^*_{12}+\eta^*_{66}\right)e_2 + \eta^*_{61}e_3 - \zeta^*_{45}e_5,$$

$$U_5 = (\overline{s}_{12}+\overline{s}_{66})e_9 + \overline{s}_{26}e_{10} + \overline{s}_{16}e_{11},$$
$$W_5 = 3\overline{\eta}_{26}e_{10} + (\overline{\eta}_{12}+2\overline{\eta}_{66})e_9 + \overline{\eta}_{16}e_{11} + \overline{\eta}_{22}e_8 - \overline{\zeta}_{45}e_5 - \overline{\zeta}_{44}e_{12},$$
$$V_5 = \overline{s}_{22}e_8 + \overline{s}_{66}e_9 + 2\overline{s}_{26}e_{10},$$
$$\Psi_5 = \eta^*_{26}e_{10} + \left(\eta^*_{12}+\eta^*_{66}\right)e_9 + \eta^*_{61}e_{11} - \zeta^*_{45}e_5,$$
$$\Phi_5 = \eta^*_{66}e_9 + 2\eta^*_{26}e_{10} + \eta^*_{22}e_8 - \zeta^*_{44}e_{12},$$
$$s_{ij} = \alpha B_{ij} + \gamma E_{ij}, \quad \overline{s}_{ij} = \beta B_{ij} + \gamma E_{ij},$$
$$\overline{\eta}_{ij} = D_{ij}\alpha\beta + \gamma F_{ij}(\beta+\alpha) + H_{ij}\gamma^2, \qquad i,j = 1,2,6,$$
$$\eta_{ij} = D_{ij}\alpha^2 + 2\alpha\gamma F_{ij} + H_{ij}\gamma^2,$$
$$\eta^*_{ij} = D_{ij}\beta^2 + 2\gamma\beta F_{ij} + H_{ij}\gamma^2, \qquad i,j = 1,2,6,$$
$$\zeta_{ij} = 9F_{ij}\gamma^2 + 6\gamma(1+\alpha)D_{ij} + (1+\alpha)^2 A_{ij},$$
$$\zeta^*_{ij} = 9F_{ij}\gamma^2 + 6\gamma\beta D_{ij} + \beta^2 A_{ij}, \qquad i,j = 4,5,$$
$$\overline{\zeta}_{ij} = 9F_{ij}\gamma^2 + 3\gamma(1+\alpha+\beta)D_{ij} + \beta(1+\alpha)A_{ij}, \quad i,j = 4,5,$$
$$k_1 = (k_{22}L_3 + k_{24}L_5 + k_{25}L_7 + k_{12}L_1 + k_{23})L_3 + (k_{44}L_5 + k_{14}L_1 + k_{34} + k_{45}L_7)L_5 + (k_{11}L_1 + k_{15}L_7 + k_{13})L_1 + k_{33} + (k_{55}L_7 + k_{35})L_7,$$
$$k_2 = (k_{24}L_6 + k_{12}L_2 + k_{25}L_8 + 2k_{22}L_4)L_3 + (k_{24}L_5 + k_{25}L_7 + k_{12}L_1 + k_{23})L_4 + (k_{45}L_8 + k_{14}L_2 + 2k_{44}L_6)L_5 + (k_{13} + k_{15}L_7 + 2k_{11}L_1)L_2 + (k_{14}L_1 + k_{34} + k_{45}L_7)L_6 + (2k_{55}L_7 + k_{15}L_1 + k_{35})L_8,$$
$$k_3 = (k_{22}L_4 + k_{24}L_6 + k_{12}L_2 + k_{25}L_8)L_4 + (k_{11}L_2 + k_{14}L_6 + k_{15}L_8)L_2 + (k_{45}L_6 + k_{55}L_8)L_8 + k_{44}L_6^2 + \mu_3 e_7,$$
$$k_4 = I_1\left(L_1^2e_{12} + L_3^2e_6\right) + \left(I_3\alpha^2 + I_7\gamma^2 + 2I_5\alpha\gamma\right)(e_6+e_{12}) + \left(2I_5\beta\gamma + I_3\beta^2 + I_7\gamma^2\right)\left(L_5^2e_6 + L_7^2e_{12}\right) + 2\hat{I}_2e_5(L_1L_5 + L_3L_7) + 2\left(I_5\beta\gamma + I_7\gamma^2 + I_3\alpha\beta + I_5\alpha\gamma\right)(L_5e_6 + L_7e_{12}) + 2\overline{I}_2e_5(L_3 + L_1),$$
$$k_5 = 2I_1(L_1L_2e_{12} + L_3L_4e_6) + 2\left(2I_5\beta\gamma + I_3\beta^2 + I_7\gamma^2\right)(L_5L_6e_6 + L_7L_8e_{12}) + 2\overline{I}_2e_5(L_2 + L_1L_6 + L_4) + 2\left(I_5\beta\gamma + I_7\gamma^2 + I_3\alpha\beta + I_5\alpha\gamma\right)(L_8e_{12} + L_6e_6) + 2\hat{I}_2e_5(L_2L_5 + L_4L_7 + L_3L_8),$$
$$k_6 = I_1\left(L_4^2e_6 + L_2^2e_{12}\right) + \left(2I_5\beta\gamma + I_3\beta^2 + I_7\gamma^2\right)\left(L_8^2e_{12} + L_6^2e_6\right) + 2\hat{I}_2e_5(L_4L_8 + L_2L_6),$$
$$k_{11} = \tfrac{1}{2}(A_{11}e_{18} + 2A_{16}e_{10} + A_{66}e_{19}),$$
$$k_{12} = A_{16}e_3 + A_{66}e_{14} + A_{12}e_{18} + A_{26}e_{10},$$
$$k_{13} = e_3s_{11} + (s_{12}+2s_{66})e_{10} + (e_{14}+2e_{18})s_{16} + e_{19}s_{26},$$
$$k_{14} = e_3\overline{s}_{11} + (e_{14}+e_{18})\overline{s}_{16} + e_{10}\overline{s}_{66},$$
$$k_{15} = e_{10}(\overline{s}_{12}+\overline{s}_{66}) + e_{18}\overline{s}_{16} + e_{19}\overline{s}_{26},$$
$$k_{22} = A_{26}e_3 + \tfrac{1}{2}(A_{22}e_{18} + A_{66}e_{20}),$$
$$k_{23} = e_{10}s_{22} + e_3(s_{12}+2s_{66}) + (e_{14}+2e_{18})s_{26} + e_{20}s_{16},$$
$$k_{24} = e_3(\overline{s}_{12}+\overline{s}_{66}) + e_{20}\overline{s}_{16} + e_{18}\overline{s}_{26},$$
$$k_{25} = e_{10}\overline{s}_{22} + (e_{14}+e_{18})\overline{s}_{26} + e_3\overline{s}_{66},$$
$$k_{33} = 2(e_3\eta_{16} + e_{10}\eta_{26} + e_{18}\eta_{66}) + \tfrac{1}{2}(e_{20}\eta_{11} + 2e_{14}\eta_{12} + e_{19}\eta_{22} + e_{12}\zeta_{44} + 2e_5\zeta_{45} + e_6\zeta_{55}),$$
$$k_{34} = e_{20}\overline{\eta}_{11} + e_{14}\overline{\eta}_{12} + 3e_3\overline{\eta}_{16} + e_{10}\overline{\eta}_{26} + 2e_{18}\overline{\eta}_{66} + e_5\overline{\zeta}_{45} + e_6\overline{\zeta}_{55},$$
$$k_{35} = e_{19}\overline{\eta}_{22} + e_{14}\overline{\eta}_{12} + e_3\overline{\eta}_{16} + 3e_{10}\overline{\eta}_{26} + 2e_{18}\overline{\eta}_{66} + e_5\overline{\zeta}_{45} + e_{12}\overline{\zeta}_{44},$$
$$k_{44} = \tfrac{1}{2}\left(e_{20}\eta^*_{11} + 2e_3\eta^*_{16} + e_{18}\eta^*_{66} + e_6\zeta^*_{55}\right),$$
$$k_{45} = e_{14}\eta^*_{12} + e_3\eta^*_{16} + e_{10}\eta^*_{26} + e_{18}\eta^*_{66} + e_5\zeta^*_{45},$$
$$k_{55} = \tfrac{1}{2}\left(e_{19}\eta^*_{22} + 2e_{10}\eta^*_{26} + e_{18}\eta^*_{66} + e_{12}\zeta^*_{44}\right),$$
$$L_1 = -\omega^2\Delta_{11}, \quad L_3 = -\omega^2\Delta_{21}, \quad L_5 = -\omega^2\Delta_{41},$$
$$L_7 = -\omega^2\Delta_{51}, \quad L_2 = l\Delta_{11} + \Delta_{12}, \quad L_4 = l\Delta_{21} + \Delta_{22},$$
$$L_6 = l\Delta_{41} + \Delta_{42}, \quad L_8 = l\Delta_{51} + \Delta_{52},$$
$$\Delta_0 = \begin{vmatrix} U_{1mn} & V_{1mn} & \Psi_{1mn} & \Phi_{1mn} \\ U_{2mn} & V_{2mn} & \Psi_{2mn} & \Phi_{2mn} \\ U_{4mn} & V_{4mn} & \Psi_{4mn} & \Phi_{4mn} \\ U_{5mn} & V_{5mn} & \Psi_{5mn} & \Phi_{5mn} \end{vmatrix},$$

$$\Delta_{11} = \begin{vmatrix} W_{1mn} & V_{1mn} & \Psi_{1mn} & \Phi_{1mn} \\ W_{2mn} & V_{2mn} & \Psi_{2mn} & \Phi_{2mn} \\ W_{4mn} & V_{4mn} & \Psi_{4mn} & \Phi_{4mn} \\ W_{5mn} & V_{5mn} & \Psi_{5mn} & \Phi_{5mn} \end{vmatrix},$$

$$\Delta_{12} = \begin{vmatrix} 0 & V_{1mn} & \Psi_{1mn} & \Phi_{1mn} \\ 0 & V_{2mn} & \Psi_{2mn} & \Phi_{2mn} \\ \overline{W}_{2mn} & V_{4mn} & \Psi_{4mn} & \Phi_{4mn} \\ \overline{W}_{3mn} & V_{5mn} & \Psi_{5mn} & \Phi_{5mn} \end{vmatrix},$$

$$\Delta_{21} = \begin{vmatrix} U_{1mn} & W_{1mn} & \Psi_{1mn} & \Phi_{1mn} \\ U_{2mn} & W_{2mn} & \Psi_{2mn} & \Phi_{2mn} \\ U_{4mn} & W_{4mn} & \Psi_{4mn} & \Phi_{4mn} \\ U_{5mn} & W_{5mn} & \Psi_{5mn} & \Phi_{5mn} \end{vmatrix},$$

$$\Delta_{22} = \begin{vmatrix} U_{1mn} & 0 & \Psi_{1mn} & \Phi_{1mn} \\ U_{2mn} & 0 & \Psi_{2mn} & \Phi_{2mn} \\ U_{4mn} & \overline{W}_{2mn} & \Psi_{4mn} & \Phi_{4mn} \\ U_{5mn} & \overline{W}_{3mn} & \Psi_{5mn} & \Phi_{5mn} \end{vmatrix},$$

$$\Delta_{41} = \begin{vmatrix} U_{1mn} & V_{1mn} & W_{1mn} & \Phi_{1mn} \\ U_{2mn} & V_{2mn} & W_{2mn} & \Phi_{2mn} \\ U_{4mn} & V_{4mn} & W_{4mn} & \Phi_{4mn} \\ U_{5mn} & V_{5mn} & W_{5mn} & \Phi_{5mn} \end{vmatrix},$$

$$\Delta_{42} = \begin{vmatrix} U_{1mn} & V_{1mn} & 0 & \Phi_{1mn} \\ U_{2mn} & V_{2mn} & 0 & \Phi_{2mn} \\ U_{4mn} & V_{4mn} & \overline{W}_{2mn} & \Phi_{4mn} \\ U_{5mn} & V_{5mn} & \overline{W}_{3mn} & \Phi_{5mn} \end{vmatrix},$$

$$\Delta_{51} = \begin{vmatrix} U_{1mn} & V_{1mn} & \Psi_{1mn} & W_{1mn} \\ U_{2mn} & V_{2mn} & \Psi_{2mn} & W_{2mn} \\ U_{4mn} & V_{4mn} & \Psi_{4mn} & W_{4mn} \\ U_{5mn} & V_{5mn} & \Psi_{5mn} & W_{5mn} \end{vmatrix},$$

$$\Delta_{52} = \begin{vmatrix} U_{1mn} & V_{1mn} & \Psi_{1mn} & 0 \\ U_{2mn} & V_{2mn} & \Psi_{2mn} & 0 \\ U_{4mn} & V_{4mn} & \Psi_{4mn} & \overline{W}_{2mn} \\ U_{5mn} & V_{5mn} & \Psi_{5mn} & \overline{W}_{3mn} \end{vmatrix},$$

$$\Delta_{mn} = \Delta_{11}U_3 + \Delta_{21}V_3 + \Delta_{41}\Psi_3 + \Delta_{51}\Phi_3 - \Delta_0 W_3,$$

$$\Delta_{1mn} = \Delta_0\overline{W}_1 - \Delta_{12}U_3 - \Delta_{22}V_3 - \Delta_{42}\Psi_3 - \Delta_{52}\Phi_3,$$

$$a_1 = -4k_6 l^6, \quad a_2 = 4k_6 l^4(l + 2k_3^3\omega^2),$$

$$a_3 = k_2k_6l^2(k_2l^2 + 4k_3\omega^2 l + 4k_3^2\omega^4),$$

$$a_4 = -4k_3^3(2k_3\omega^2 + k_2 l), \quad a_5 = k_3^3(4k_3k_1 - k_2^2),$$

$$a_6 = -2k_1, \quad a_7 = -4k_3k_6l^4,$$

$$a_8 = 21(k_3^2k_5l^3 - k_2k_3k_6l^3 - k_3^3l^2), \quad a_9 = 2k_3^2k_6l^2,$$

$$a_{10} = 2k_3^2(2k_3^2 - k_3 l + k_6 l),$$

$$a_{11} = \tfrac{1}{2}k_3(4k_3^3k_4 + k_2^2k_6 - 2k_2k_3^2k_5), \quad a_{12} = 4k_3^4,$$

$$a_{13} = 2k_6l^5(k_2 + 2k_3l\omega^2), \quad a_{14} = -a_{1,},$$

$$a_{15} = k_3^2 a_7,$$

$$a_{16} = 2k_3^2l^2(k_5l - 2k_2k_6l - 2k_6\omega^2 - 2k_3),$$

$$a_{17} = e_2k_3^2k_5l^2 + 2k_3^3k_5l\omega^2 - k_2^2k_3k_6l^2 - 2k_2k_3^2l\omega^2k_6 - 4k_3^4\omega^2 - 2k_2k_3^3l.$$

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