Research Project

On Generation of Certain Simple Groups by Harada-Norton and Tits

by

Mohammed Ali Faya Ibrahim
&
Faryad Ali

Department of Mathematics, Faculty of Science
King Khalid University, Abha,
South Africa.

June 2005

Groups play a central role in just about all branches of Mathematics and continue to be a very active area of research. Computational group theory is a subject of great topical interest and has many applications in Mathematics and other sciences. By the classification of finite simple groups in 1981, it is now known that every finite simple group is either cyclic of prime order, an Alternating group of degree n ($n \geq 5$), a simple group of *Lie* type, or one of 26 sporadic simple groups.

There can be no question that the modern computer offers a powerful resource for supporting research and teaching of mathematics. The power of this resource, however, can be made easily accessible to academics and students through appropriate software-package as computational tools. The widespread availability of such software in our present times has a definite impact on the research we do as well as the contents and presentation of the mathematics we teach, as they make some topics redundant and others necessary.

The growth of computational group theory over the last thirty years has stimulated the demand for software, which would permit the exploitation of these techniques by the wider group theory community. The development in the late seventies and early eighties of machine independent software implementing particular group theory algorithms presented the first step towards meeting this demand. Pure mathematicians were stirred by, but in most cases also confined by the information that was produced by group theoretic software for their special research problem and hampered by the uneasy feeling that one was dealing with black boxes of uncontrollable reliability. However, the last years have seen a rapid spread of the interest in the understanding, design and even implementation of group theoretical algorithms. These are gradually becoming accepted both as a standard tool for a working group theorist, like certain methods of proof, and as worthwhile objects of study, like connections between notions expressed in theorems.

The software-packages **GAP** and **MAGMA** were started as an attempt to meet this interest. These packages relieves the user from unwanted technical cores and assist him/her in the programming, thus supporting invention and implementation of the new algorithms as well as experimenting with them.

Present Research

A group G is said to be 2-*generated* if it can be generated by two suitable elements.

It is well-known that every finite simple group is 2-generated. This has been known for a long time (*cf.* Miller [**?**]) in the case of the alternating groups. Explicit generators for the alternating groups A_n, with $n \geq 5$ (*cf.* Aschbacher-Guralnick, [**?**]) are:

$$a = (1,2)(n-1,n) \quad \text{and} \quad b = (1,2,\ldots,n-1), \quad \text{if } n \text{ is even}$$

$$a = (1,n)(2,n-1) \quad \text{and} \quad b = (1,2,\ldots,n-2), \quad \text{if } n \text{ is odd.}$$

For the groups of Lie type, Steinberg [**?**] provided a unified treatment for the 2-generation of the Chevalley groups and the Twisted groups. Steinberg's construction of a generating pair exploits the basic structure of a group of Lie type. Before this the 2-generation of certain families of Lie-groups were known (eg. $PSL(n,F)$ and $Sp(n,F)$). For the sporadic simple groups we have the following result.

Theorem 0.0.1 *(Aschbacher-Guralnick [**?**]) Every sporadic simple group can be generated by an involution and another suitable element.* □

Aschbacher and Guralnick were primarily concerned with applications to cohomology. They prove: *Let G be a finite group acting faithfully and irreducibly on a vector space V over the prime field $GF(p)$. Then $|H^1(G,V)| < |V|$, where $H^1(G,V)$ is the first cohomology group of G on V.* The 2-generation of the simple groups come into play in the following way. First it is proved that if G is generated by d elements then, $|H^1(G,V)| < |V|^{d-1}$, then a reduction to the case G simple is accomplished and 2-generation gives the required result.

In view of applications (as noted above), it is often important to exhibit generating pairs of some special kind, such as:

- generators carrying a geometric meaning,
- generators of some prescribed order,
- generators that offer an economical presentation of the group.

For this purpose, more subtle and detailed techniques are required. We now examine such instances.

1. Genus action: A group G is said to be $(n_1, \ldots, n_h)$-generated if G is a quotient of the group

$$\Gamma = \langle x_1, \ldots, x_h \mid {x_1}^{n_1} = {x_2}^{n_2} = \cdots = {x_h}^{n_h} = x_1 x_2 \cdots x_h = 1_G \rangle.$$

In the case where $h = 3, 4$ we call Γ a triangular group $T(n_1, n_2, n_3)$ and quadrangular group $Q(n_1, \ldots, n_4)$, respectively. The *genus* $g(G)$ of a finite group G is defined to be the smallest integer g such that some Cayley graph of G is embedded on a Riemann surface S_g with genus g. The action of groups on Riemann surfaces seeks a geometric representation theory of finite groups as automorphism groups of Riemann surfaces. The genus action plays the role of an irreducible representations in this theory.

Let Γ be a $(n_1, \ldots, n_h)$-generated finite group and let $\mathcal{H}^2$ be the hyperbolic plane. If G is a homomorphic image of Γ, then the short exact sequence

$$1_\Delta \to \Delta \to \Gamma \to G \to 1_G.$$

gives rise to an orbit space $S_g = \mathcal{H}^2/\Delta$ in the natural way of the structure of a Riemann surface on which G acts faithfully as a group of conformal mappings. Moreover, the regular branch covering $\mathcal{H}^2/\Delta \to \mathcal{H}^2/\Gamma$ has branch point orders $n_1, \ldots, n_h$. The genus of $\mathcal{H}^2/\Delta$, hence of G, can be calculated from the genus of $\mathcal{H}^2/\Gamma$ by the well-known Riemann-Hurwitz formula

$$g(\mathcal{H}^2/\Delta) = 1 + \frac{|G|}{2}[g(\mathcal{H}^2/\Gamma) - 2 + \sum_{I=1}^{h}(1 - 1/n_i)]$$

We now restrict ourselves to finite simple groups. It is conjectured that every finite non-abelian finite simple group can be generated by an involution and another suitable element, that is, $(2, s, t)$-generated. The validity of this conjecture will simplify the calculation of the genus of finite simple groups as follows.

Proposition 0.0.2 *(Woldar [22]) Let G be a finite non-abelian $(2, s, t)$-generated group and S a Riemann surface of least genus on which G acts. Then $S/G = \mathcal{S}^2$ (the 2-sphere) and the branch covering $\pi : S \to S/G$ has either* 3 *or* 4 *branch points.* □

Thus the genus action of the $(2, s, t)$-generated finite groups arise from the short exact sequence $1_{Delta} \to \Delta \to \Gamma \to G \to 1_G$, where Γ is either a triangular group or a quadrangular group. As a consequence of the Riemann-Hurwitz equation the genus of a $(2, s, t)$-generated group G is given by

$$g(G) = 1 + \frac{|G|}{2}M,$$

where $M = 1 - 1/l - 1/m - 1/n$ or $M = 2 - 1/u - 1/v - 1/w - 1/x$, depending, respectively, on whether $\Gamma = T(l, m, n)$ or $Q(u, v, w, x)$ in the genus action. Thus the genus problem of the these groups is reduced to a problem on generations. With this in mind Moori [14] posed the following problem.

(1) *Let G be a finite simple group such that l, m, n are divisors of $|G|$ with $1/l + 1/m + 1/n < 1$. Is G a (l, m, n)-generated group?*

2. Spread: Let r be any positive integer. A finite non-abelian group G is said to have *spread* r, if for every set $\{x_1, x_2, \ldots, x_r\}$ of distinct non-trivial elements of G, there exists a complementary $y \in G$ such that $G = \langle x_i, y\rangle$ for all i. Let G be a finite group and nX a conjugacy class of G. The group G is called *nX-complementary generated* if, given an arbitrary $x \in G$, complementary y can always be chosen from the conjugacy class nX. Woldar proved that every sporadic simple group G is pA-complementary generated, where p is the largest prime divisor of $|G|$. In an attempt to further the theory on nX-complementary generations we pose the following problem.

(2) *Find all conjugacy classes nX of a finite simple group G such that G is nX-Complementary generated.*

3. Generation by Conjugate Elements. There has recently been some interest in generation of simple groups by their conjugate involutions. It is well known that sporadic simple groups are generated by three conjugate involutions (see [6]). If a group $G = \langle a, b\rangle$ is perfect and $a^2 = b^3 = 1$ then clearly G is generated by three conjugate involutions a, a^b and a^{b^2} (see [7]). Moori [15] proved that the Fischer group Fi_{22} can be generated by three conjugate involutions. The work of Liebeck and Shalev [12] show that all but finitely many classical groups can be generated by three involutions. The generation of a simple group by its conjugate elements in this context is of some interest.

Suppose that G is a finite group and $X \subseteq G$. We denote the rank of X in G by $rank(G{:}X)$, the minimum number of elements of X generating G. This paper focuses on the determination of $rank(G{:}X)$ where X is a conjugacy class of G and G is a sporadic simple group.

Moori in [13], [14] and [15] proved that $rank(Fi_{22}{:}2A) \in \{5, 6\}$ and $rank(Fi_{22}{:}2B) = rank(Fi_{22}{:}2C) = 3$ where $2A, 2B$ and $2C$ are the conjugacy classes of involutions of the smallest Fischer group Fi_{22} as presented in the *ATLAS* [4]. The work of Hall and Soicher [9] show that $rank(Fi_{22}{:}2A) = 6$. Moori in [16] determined the ranks of the *Janko* groups J_1, J_2 and J_3. More recently, in a series of papers [1, 2, 3, 10], we investigated the ranks for the sporadic group HS, McL, Co_1 Co_2, Co_3, Ru, Suz and Th.

Proposed Research

The central role-played in the representation theory of finite groups and algebraic groups has progressively become clear during the past two decades. There has been great progress in purely combinatorial theory of finite groups. The character tables of all the maximal subgroups of simple groups are not yet known. I would like to apply the method of Fischer-Clifford matrices to construct the character tables of certain group extensions related to sporadic simple groups and finite groups of Lie type. This method has the advantage that character tables of pretty complicated groups can be computed because the arithmetical and combinatorial properties of the matrices involved are very powerful. For our computations, in addition to **GAP** and **MAGMA** systems, we would like to investigate the possibility of using **MeatAxe**, a computer algebra package freely available in the GAP.

For most group theorists studying generations, the main aim is not just to give generators, but to offer economical presentations for the groups in question. For the finite non-abelian simple groups 2-generations is an ideal starting point. Solutions to the three problems posed above will provide us with a pool of generations pairs (together with some relations) which may in time be extended to (abstract) presentations of the groups.

In this project we will focus on problems (1), (2) and (3) for the Harada-Norton sporadic simple group HN and for the Tits group Tits. We will provide a complete answer to problem for these groups.

Expected Outcome of the Proposed Research

It is expected that the research in this project will bring fruitful results of great significance, which will be publishable in internationally recognized journals. As a result of this research plan, we would like to gain a better understanding of the interaction between the character theory of sporadic groups and the generation type problems relating to these simple groups. This project will give us a more clear direction for our future research.

Bibliography

[1] F. Ali and M. A. F. Ibrahim, *On the ranks of HS and McL*, Utilitas Math., to appear.

[2] F. Ali and M. A. F. Ibrahim, *On the ranks of Conway groups Co_2 and Co_3*, J. Algebra Appl., to appear.

[3] F. Ali and M. A. F. Ibrahim, *On the ranks of Conway group Co_1*, Proc. Japan Acad. Ser. A, to appear.

[4] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, and R. A. Wilson. *An Atlas of Finite Groups*, Oxford University Press, 1985.

[5] M. D. E. Conder, R. A. Wilson and A. J. Woldar, *The symmetric genus of sporadic groups*, Proc. Amer. Math. Soc. **116** (1992), 653–663.

[6] F. Dalla Volta, *Gruppi sporadici generati da tre involzionni*, RILS **119** (1985), 65-87.

[7] L. Di Martino and C. Tamburini, *2-Generation of finite simple groups and some related topics*, Generators and Relations in Groups and Geometry, A. Barlotti et al., Kluwer Acad. Publ., New York (1991), 195 – 233.

[8] S. Ganief and J. Moori, *Generating pairs for the Conway groups Co_2 and Co_3*, J. Group Theory **1** (1998), 237-256.

[9] J. I. Hall and L. H. Soicher, *Presentations of some 3-transposition groups*, Comm. Algebra **23** (1995), 2517-2559.

[10] M. A. F. Ibraheem, *On the ranks of certain sporadic simple groups by Suzuki, Thompson and Rudvalis*, Algebras, Groups and Geometries, to appear.

[11] I. M. Isaacs, *Character Theory of Finite Groups*, Dover, New York, 1994.

[12] M. W. Liebeck and A. Shalev, *Classical groups, probabilistic methods and* $(2,3)$*-generation problem*, Annals of Math, **144** (1996), 77-125.

[13] J. Moori, *Generating sets for* F_{22} *and its automorphism group*, J. Algebra **159** (1993), 488–499.

[14] J. Moori, *Subgroups of* 3*-transposition groups generated by four* 3*-transpositions*, Quaest. Math. **17** (1994), 83–94.

[15] J. Moori, *On the ranks of the Fischer group* F_{22}, Math. Japonica , **43**(2) (1996), 365–367.

[16] J. Moori, *On the ranks of Janko groups* J_1*,* J_2 *and* J_3*,* Article presented at the 41st annual congress of South African Mathematical Society, RAU 1998.

[17] S. P. Norton, *F and other simple group*, PhD thesis, Cambridge, 1975.

[18] L. L. Scott, *Matrices and cohomology*, Ann. Math. **105**(3) (1977), 473–492.

[19] The GAP Group, *GAP - Groups, Algorithms and Programming, Version 4.3* , Aachen, St Andrews, 2003, (`http://www.gap-system.org`).

[20] R. A. Wilson et al., A world-wide-web *Atlas of Group Representations*, (`http://web.mat.bham.ac.uk/atlas`).

[21] A. J. Woldar, *On Hurwitz generation and genus actions of sporadic groups*, Illinois J. Math. **33**(3) (1989), 416–437.

[22] A. J. Woldar, *Representing* M_{11}*,* M_{12}*,* M_{22} *and* M_{23} *on surfaces of least genus*, Comm. Algebra **18** (1990), 15–86.