# On the generation of the Conway groups by conjugate involutions*

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**Abstract**

Let $G$ be a finite group generated by conjugate involutions, and let $i(G) = min\{|X|\}$, where $X$ runs over the sets of conjugate involutions generating $G$. Of course $i(G) \leq 2$ implies $G$ is cyclic or dihedral. However, the problem of determining those $G$ for which $i(G) > 2$ is much more intricate. In this note, we prove that $i(G) \leq 4$, where $G$ is one of the Conway's sporadic simple group. The computations were carried out using the computer algebra system $\mathbb{GAP}$ [15].

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## 1 Introduction

It is well known that sporadic simple groups are generated by three conjugate involutions (see [7]). Recently there has been considerable interest in generation of simple groups by their conjugate involutions. Moori [14] proved that the Fischer group $Fi_{22}$ can be generated by three conjugate involutions. The work of Liebeck and Shalev [13] show that all but finitely many classical groups can be generated by three involutions. Moori and Ganief in [12] determined the generating pairs for the Conway groups $Co_2$ and $Co_3$. Darafsheh, Ashrafi and Moghani in [8, 9, 10] computed the $(p, q, r)$ and $nX$-complementary generations for the largest Conway group $Co_1$, while recently Bates and Rowley in [5] determined the suborbits of Conway's largest simple group in its conjugation action on each of its three conjugacy classes of involutions. More recently, the authors in [2, 3] computed the ranks for the Conway groups. In this note, we compute the minimal generating involution sets for the Conway's sporadic simple groups.

Let $G$ be a finite group generated by conjugate involutions, and let $i(G) = min\{|X|\}$, where $X$ runs over the sets of conjugate involutions generating $G$. Since $i(G) \leq 2$ implies $G$ is cyclic or dihedral we are interested in determining those $G$ for which $i(G) > 2$. In this note, we show that $i(G) \in \{3, 4\}$, where $G$ is one of the Conway's sporadic simple group.

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Throughout this paper we use the same terminology and notation as in [1, 2, 3, 4] and [12]. In particular, if $G$ is a finite group, $C_1, C_2, \cdots, C_k$ are the conjugacy classes of its elements and $g_k$ is a fixed representative of $C_k$, then $\Delta_G(C_1, C_2, \cdots, C_k)$ denotes the number of distinct tuples $(g_1, g_2, \cdots, g_{k-1}) \in (C_1 \times C_2 \times \cdots \times C_{k-1})$ such that $g_1 g_2 \cdots g_{k-1} = g_k$. It is well known that $\Delta_G(C_1, C_2, \cdots, C_k)$ is the structure constant of $G$ for the conjugacy classes $C_1, C_2, \cdots, C_k$ and can be computed from the character table of $G$ (see [**?**], p.45 ) by the following formula

$$\Delta_G(C_1, C_2, \cdots, C_k) \quad = \quad \frac{|C_1||C_2|\cdots|C_{k-1}|}{|G|} \times \sum_{i=1}^{m} \frac{\chi_i(g_1)\chi_i(g_2)\cdots\chi_i(g_{k-1})\overline{\chi_i(g_k)}}{[\chi_i(1_G)]^{k-2}}$$

where $\chi_1, \chi_2, \cdots, \chi_m$ are the irreducible complex characters of $G$. Also, $\Delta_G^*(C_1, C_2, \cdots, C_k)$ denotes the number of distinct tuples $(g_1, g_2, \cdots, g_{k-1}) \in (C_1 \times C_2 \times \cdots \times C_{k-1})$ such that $g_1 g_2 \cdots g_{k-1} = g_k$ and $G = \langle g_1, g_2, \cdots, g_{k-1} \rangle$. If $\Delta_G^*(C_1, C_2, \cdots, C_k) > 0$, then we say that $G$ is $(C_1, C_2, \cdots, C_k)$-generated. If $H$ any subgroup of $G$ containing the fixed element $g_k \in C_k$, then $\Sigma_H(C_1, C_2, \cdots, C_{k-1}, C_k)$ denotes the number of distinct tuples $(g_1, g_2, \cdots, g_{k-1}) \in (C_1 \times C_2 \times \cdots \times C_{k-1})$ such that $g_1 g_2 \cdots g_{k-1} = g_k$ and $\langle g_1, g_2, \cdots, g_{k-1} \rangle \leq H$ where $\Sigma_H(C_1, C_2, \cdots, C_k)$ is obtained by summing the structure constants $\Delta_H(c_1, c_2, \cdots, c_k)$ of $H$ over all $H$-conjugacy classes $c_1, c_2, \cdots, c_{k-1}$ satisfying $c_i \subseteq H \cap C_i$ for $1 \leq i \leq k-1$.

The number of conjugates of a given subgroup $H$ of $G$ containing a fixed element $c$ is given by $h = \chi_{N_G(H)}(c)$, where $\chi_{N_G(H)}$ is the permutation character of $G$ with action on the conjugates of $H$.

## 2 The largest Conway group $Co_1$

The Conway group $Co_1$ is a sporadic simple group of order

$$4,157,776,806,543,360,000 = 2^{21}.3^9.5^4.11.13.23.$$

The subgroup structure of $Co_1$ is discussed in Wilson [17]. The group $Co_1$ has exactly 22 conjugacy classes of maximal subgroups as listed in Wilson [17]. $Co_1$ has 101 conjugacy classes of its elements. It has precisely three classes of involutions, namely $2A$, $2B$ and $2C$ as represented in the $\mathbb{ATLAS}$ [6]. For basic properties of $Co_1$ and information on its maximal subgroups the reader is referred to [6], [8] and [17].

**Lemma 1** *The group $Co_1$ can be generated by three conjugate involutions $a, b, c \in 2X$ for all $X \in \{A, B, C\}$ such that $abc \in 11A$.*

**Proof:** The case that $X = A$ has been proved in [3] as Lemma 2.1.

Next we consider the case $X = B$. We compute using $\mathbb{GAP}$ that $\Delta_{Co_1}(2B, 2B, 2B, 11A) = 2073535860$. In $Co_1$ we have only two maximal subgroups, up to isomorphism, with orders divisible by 13 and having non-empty intersection with classes $2B$ and $11A$, namely, $H_1 \cong 3.Suz.2$ and $H_2 \cong 2^{11}{:}M_{24}$. We also have $\Sigma_{H_1}(2B, 2B, 2B, 11A) = 21853689$. A fixed element of order 11 in $Co_1$ lies in a unique conjugate copy of $H_1$. Hence $H_1$ contributes $1 \times 21853689 = 21853689$ to the number $\Delta_{Co_1}(2B, 2B, 2B, 11A)$.

Similarly, we obtain that $\Sigma_{H_2}(2B,2B,2B,11A) = 2398704$ and a fixed element of order 11 in $Co_1$ lies in precisely three conjugate subgroups of $H_2$. This mean that $H_2$ contributes $3 \times 2398704 = 7196112$ to the number $\Delta_{Co_1}(2B,2B,2B,11A)$. Since

$$\Delta^*_{Co_1}(2B,2B,2B,11A) \geq 2073535860 - 21853689 - 7196112 > 0,$$

the group $Co_1$ is $(2B,2B,2B,11A)$-generated.

Finally, consider the case $X = C$. The only maximal subgroups of $Co_1$ that may contain $(2C,2C,2C,11A)$-generated subgroups are isomorphic to $Co_2$, $3.Suz.2$, $2^{11}{:}M_{24}$, $Co_3$, $U_6(2).3.2$ and $3^6{:}2M_{12}$. We calculate that $\Sigma_{Co_2}(2C,2C,2C,11A) = 555389032$, $\Sigma_{3.Suz.2}(2C,2C,2C,11A) = 0$, $\Sigma_{2^{11}:M_{24}}(2C,2C,2C,11A) = 21845824$, $\Sigma_{Co_3}(2C,2C,2C,11A) = 35424928$, $\Sigma_{U_6(2).3.2}(2C,2C,2C,11A) = 39785526$, $\Sigma_{3^6:2M_{12}}(2C,2C,2C,11A) = 176418$ and we obtain that

$$\begin{aligned}\Delta^*_{Co_1}(2C,2C,2C,11A) \geq\ & \Delta_{Co_1}(2C,2C,2C,11A) - 6\Sigma_{Co_2}(2C,2C,2C,11A)\\ & -3\Sigma_{2^{11}:M_{24}}(2C,2C,2C,11A) - 6\Sigma_{Co_3}(2C,2C,2C,11A)\\ & -2\Sigma_{U_6(2).3.2}(2C,2C,2C,11A) - 2\Sigma_{3^6:2M_{12}}(2C,2C,2C,11A) > 0\end{aligned}$$

Hence the group $Co_1$ is $(2C,2C,2C,11A)$-generated. □

**Lemma 2** *Let $Co_1$ be the Conway's largest sporadic simple group then $i(Co_1) = 3$.*

**Proof:** In the above lemma we proved that $Co_1$ can be generated by three conjugate involutions from each conjugacy class of involution $2A$, $2B$ and $2C$. Therefore $i(G) \leq 3$. Since $i(G) = 2$ is not possible, the result follows.

## 3 The Conway group $Co_2$

The Conway group $Co_2$ is a sporadic simple group of order $2^{18}.3^6.5^3.7.11.23$ with 11 conjugacy classes of maximal subgroups. It has 60 conjugacy classes of its elements including three conjugacy classes of involutions, namely $2A, 2B$ and $2C$. The group $Co_2$ acts primitively on a set $\Omega$ of 2300 points. The point stabilizer of this action is isomorphic to $U_6(2){:}2$ and the orbits have length 1, 891 and 1408. The permutation character of $Co_2$ on the cosets of $U_6(2){:}2$ is given by $\chi_{U_6(2):2} = \underline{1a} + \underline{275a} + \underline{2024a}$. For basic properties of $Co_2$ and computational techniques, the reader is encouraged to consult [2], [**?**], [12] and [17].

**Lemma 3** *([2]) The group $Co_2$ can not be generated by three conjugate involutions from its $2A$ conjugacy class.*

**Lemma 4** *Let $Co_2$ be the Conway's second largest sporadic simple group. Then $i(Co_2) = 4$ for the conjugacy class $2A$ of $Co_2$.*

**Proof:** We compute that the structure constant $\Delta_{Co_2}(2A, 2A, 2A, 2A, 23A) = 17836822$. The only maximal subgroup of $Co_2$ which has order divisible by 23 is isomorphic to $M_{23}$. However, $2A \cap M_{23} = \emptyset$. Thus we have

$$\Delta^*_{Co_2}(2A, 2A, 2A, 2A, 23A) = \Delta_{Co_2}(2A, 2A, 2A, 2A, 23A) > 0.$$

Thus $Co_2$ can be generated by four conjugate involutions from the conjugacy class $2A$. □

Next we compute the minimal generating sets for the classes $2B$ and $2C$ of $Co_2$.

**Lemma 5** *The Conway group $Co_2$ is $(2X, 2X, 2X, 23A)$-generated for $X \in \{B, C\}$.*

**Proof:** We calculate that $\Delta_{Co_2}(2B, 2B, 2B, 23A) = 12696$ and $\Delta_{Co_2}(2C, 2C, 2C, 23A) = 549387660$. The only maximal subgroups of $Co_2$ which can have $(2X, 2X, 2X, 23A)$-generated proper subgroups is isomorphic to $M_{23}$. However, the $2B$ and $2C$ classes of $Co_2$ does not meet $M_{23}$. That is, $2B \cap M_{23} = \emptyset = 2C \cap M_{23}$. Thus, no maximal subgroup and hence no proper subgroup of $Co_2$ is $(2X, 2X, 2X, 23A)$-generated where $X \in \{B, C\}$. We obtain that

$$\Delta^*_{Co_2}(2X, 2X, 2X, 23A) = \Delta_{Co_2}(2X, 2X, 2X, 23A) > 0.$$

Therefore, $Co_2$ is $(2X, 2X, 2X, 23A)$-generated for $X \in \{B, C\}$.

**Lemma 6** *Let $Co_2$ be the Conway's second sporadic simple group then $i(Co_2) \in \{3, 4\}$.*

**Proof:** The result is now immediate from the above three lemmas.

## 4 The smallest Conway group $Co_3$

The smallest *Conway* group $Co_3$ is a sporadic simple group of order $2^{10}.3^7.5^3.7.11.23$ with 14 conjugacy classes of maximal subgroups. The group $Co_3$ has 42 conjugacy classes of its elements. It has two conjugacy classes of involutions, namely $2A$ and $2B$. For basic properties of $Co_3$ we refer readers to [6] and [11].

**Lemma 7** *Let $Co_3$ be the smallest Conway group then $i(Co_3) = 3$.*

**Proof:** There are two conjugacy classes of involutions in $Co_3$.

The only maximal subgroup of the group $Co_3$ that may contain $(2A, 2A, 2A, 23A)$-generated proper subgroup of $Co_3$, up to isomorphism, is $M_{23}$. Further, a fixed element $z \in 23A$ is contained in a unique conjugate subgroup of $M_{23}$. A simple computation reveals that $\Delta_{Co_3}(2A, 2A, 2A, 23A) = 5290$ and $\Sigma_{M_{23}}(2A, 2A, 2A, 23A) = 3174$. Since

$$\Delta^*_{Co_3}(2A, 2A, 2A, 23A) \geq \Delta_{Co_3}(2A, 2A, 2A, 23A) - \Sigma_{M_{23}}(2A, 2A, 2A, 23A) > 0,$$

we conclude that $Co_3$ can be generated by three conjugate involutions from the $2A$ class of $Co_3$. Also, we can apply similar techniques to show that $Co_3$ can be generated by three conjugate involutions from the class $2B$ of $Co_3$. This completes the proof. □

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