On the ranks of Conway group Co_1

Dedicated to Professor Jamshid Moori on the occasion of his sixtieth birthday

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Abstract

Let G be a finite group and X a conjugacy class of G. We denote $rank(G:X)$ to be the minimum number of elements of X generating G. In the present paper we investigate the ranks of the Conway group Co_1. Computations were carried with the aid of computer algebra system $\mathbb{GAP}$ [17].

1 Introduction and Preliminaries

Let G be a finite group and $X \subseteq G$. We denote the minimum number of elements of X generating G by $rank(G:X)$. In the present paper we investigate $rank(G:X)$ where X is a conjugacy class of G and G is a sporadic simple group.

Moori in [13], [14] and [15] proved that $rank(Fi_{22}:2A) \in \{5,6\}$ and $rank(Fi_{22}:2B) = rank(Fi_{22}:2C) = 3$ where $2A$, $2B$ and $2C$ are the conjugacy classes of involutions of the smallest Fischer group Fi_{22} as represented in the $\mathbb{ATLAS}$ [4]. The work of Hall and Soicher [11] shows that $rank(Fi_{22}:2A) = 6$. Moori in [16] determined the ranks of the Janko group J_1, J_2 and J_3. Recently in [1] and [2] the authors computed the ranks of the four sporadic simple groups HS, McL, Co_2 and Co_3.

In the present article, the authors continue their study to determine the ranks of the sporadic simple groups and the problem is resolved for the Conway's largest sporadic simple groups Co_1. We determine the rank for each conjugacy class of Co_1. We prove the following result:

Theorem 2.7 . Let Co_1 be the Conway's largest sporadic simple group. Then

(a) $rank(Co_1:nX) = 3$ if $nX \in \{2A, 2B, 2C, 3A\}$.

(b) $rank(Co_1:nX) = 2$ if $nX \notin \{1A, 2A, 2B, 2C, 3A\}$.

For basic properties of Co_1, character tables of Co_1 and their maximal subgroups we use $\mathbb{ATLAS}$ [4] and $\mathbb{GAP}$ [17]. For detailed information about the computational techniques used in this talk the reader is encouraged to consult [1], [10] and [15].

Throughout this paper our notation is standard and taken mainly from [1], [2] and [10]. In particular, for a finite group G with $C_1, C_2, \ldots, C_k$ conjugacy classes of its elements and g_k a fixed representative of C_k, we denote $\Delta_G(C_1, C_2, \ldots, C_k)$ the number of distinct tuples $(g_1, g_2, \ldots, g_{k-1})$ with $g_i \in C_i$ such that $g_1 g_2 \ldots g_{k-1} = g_k$. It is well known that $\Delta_G(C_1, C_2, \ldots, C_k)$ is structure constant for the conjugacy classes $C_1, C_2, \ldots, C_k$ and can be easily computed from the character table of G (see [12], p.45) by the following formula $\Delta_G(C_1, C_2, \ldots, C_k) = \frac{|C_1||C_2|\ldots|C_{k-1}|}{|G|} \times \sum_{i=1}^{m} \frac{\chi_i(g_1)\chi_i(g_2)\ldots\chi_i(g_{k-1})\overline{\chi_i(g_k)}}{[\chi_i(1_G)]^{k-2}}$ where $\chi_1, \chi_2, \ldots, \chi_m$ are the irreducible complex

2000 *Mathematics Subject Classification.* Primary 20D08, 20F05.

Key words and phrases. Conway's group Co_1, rank, generator, sporadic group.

characters of G. Further let $\Delta_G^*(C_1, C_2, \ldots, C_k)$ denote the number of distinct tuples $(g_1, g_2, \ldots, g_{k-1})$ with $g_i \in C_i$ and $g_1 g_2 \ldots g_{k-1} = g_k$ such that $G =< g_1, g_2, \ldots, g_{k-1} >$. If $\Delta_G^*(C_1, C_2, \ldots, C_k) > 0$, then we say that G is $(C_1, C_2, \ldots, C_k)$-generated. If H is ¿a subgroup of G containing g_k and B is a conjugacy class of H such that $g_k \in B$, then $\Sigma_H(C_1, C_2, \ldots, C_{k-1}, B)$ denotes the number of distinct tuples $(g_1, g_2, \ldots, g_{k-1})$ such that $g_i \in C_i$ and $g_1 g_2 \ldots g_{k-1} = g_k$ and $\langle g_1, g_2, \ldots, g_{k-1} \rangle \leq H$.

For the description of the conjugacy classes, the character tables, permutation characters and information on the maximal subgroups readers are referred to 𝔸𝕋𝕃𝔸𝕊 [4]. A general conjugacy class of elements of order n in G is denoted by nX. For example $2A$ represents the first conjugacy class of involutions in a group G. We will use the maximal subgroups and the permutation characters of Co_1 on the conjugates (right cosets) of the maximal subgroups listed in the 𝔸𝕋𝕃𝔸𝕊 [4] extensively.

The following results will be crucial in determining the ranks of a finite group G.

Lemma 1.1. *(Moori [16]) Let G be a finite simple group such that G is (lX, mY, nZ)-generated. Then G is $(\underbrace{lX, lX, \ldots, lX}_{m-times}, (nZ)^m)$-generated.*

Corollary 1.2. *Let G be a finite simple group such that G is (lX, mY, nZ)-generated, then $rank(G : lX) \leq m$.*

Proof. The proof follows immediately from Lemma 1.1. □

Lemma 1.3. *(Conder et al. [5]) Let G be a simple $(2X, mY, nZ)$-generated group. Then G is $(mY, mY, (nZ)^2)$-generated.*

We will employ results that, in certain situations, will effectively establish non-generation. They include Scott's theorem (*cf.* [5] and [18]) and Lemma 3.3 in [21] which we state here.

Lemma 1.4. *([21]) Let G be a finite centerless group and suppose lX, mY, nZ are G-conjugacy classes for which $\Delta^*(G) = \Delta_G^*(lX, mY, nZ) < |C_G(nZ)|$. Then $\Delta^*(G) = 0$ and therefore G is not (lX, mY, nZ)-generated.*

2 Ranks of Co_1

The Conway group Co_1 is a sporadic simple group of order

$$4,157,776,806,543,360,000 = 2^{21}.3^9.5^4.11.13.23.$$

The subgroup structure of Co_1 is discussed in Wilson [19]. The group Co_1 has exactly 22 conjugacy classes of maximal subgroups as listed in Wilson [19]. Co_1 has 101 conjugacy classes of its elements. It has precisely three classes of involutions, namely $2A$, $2B$ and $2C$ as represented in the 𝔸𝕋𝕃𝔸𝕊 [4]. Co_1 acts on a 24-dimensional vector space Ω over $GF(2)$ and this action produces three orbits on the set of nonzero vectors. The point stabilizers are the groups Co_2, Co_3 and $2^{11}{:}M_{24}$ and the permutation character of Co_1 on $\Omega - \{0\}$, which is given in [6], is $\chi = 3.1a + 2.299a + 2.17250a + 3.80730a + 376740a + 644644a + 2055625a + 2417415a + 2.5494125a$, where na denotes the first irreducible character with degree n. For basic properties of Co_1 and information on its maximal subgroups the reader is referred to [4], [3], [6] and [19].

Recently Darafsheh, Arshafi and Moghani in [6], [7] and [8] established (p, q, r)-generations and nX-complementary generations of the Conway group Co_1. We will make use of these generations to determine the ranks of Co_1 in some cases.

In the following we prove that the Conway group Co_1 can be generated by three involutions.

Lemma 2.1. *The group Co_1 can be generated by three involutions $a, b, c \in 2A$ such that $abc \in 13A$.*

Proof. Using the character table of Co_1 we have $\Delta_{Co_1}(2A, 2A, 2A, 13A) = 9633$. In Co_1 we have only two maximal subgroups, up to isomorphism, with orders divisible by 13, namely, $H_1 \cong$

3.Suz.2 and $H_2 \cong (A_4 \times G_2(4))$:2. We also have $\Sigma_{H_1}(2A, 2A, 2A, 13A) = \Delta_{H_1}(2A, 2A, 2A, 13A) = 1521$. A fixed element of order 13 in Co_1 lies in four conjugates of H_1. Hence H_1 contributes $4 \times 1521 = 6084$ to the number $\Delta_{Co_1}(2A, 2A, 2A, 13A)$. Similarly, we compute that $\Sigma_{H_2}(2A, 2A, 2A, 13A) = \Delta_{H_2}(2A, 2A, 2A, 13A) = 169$ and a fixed element of order 13 in Co_1 lies in a unique conjugate of H_2. This mean that H_2 contributes $1 \times 169 = 169$ to the number $\Delta_{Co_1}(2A, 2A, 2A, 13A)$. Since

$$\Delta^*_{Co_1}(2A, 2A, 2A, 13A) \geq 9633 - 6084 - 169 > 0,$$

the group Co_1 is $(2A, 2A, 2A, 13A)$-generated. □

Lemma 2.2. *Let Co_1 be the Conway's largest sporadic group Co_1 then $rank(Co_1 : 2X) = 3$ where $X \in \{A, B, C\}$.*

Proof. We proved in the previous lemma that Co_1 is $(2A, 2A, 2A, 13A)$-generated and so $rank(Co_1 : 2A) \leq 3$ but $rank(Co_1 : 2A) = 2$ is not possible, because if $\langle x, y\rangle = Co_1$ for some $x, y \in 2A$ then $Co_1 \cong D_{2n}$ with $o(xy) = n$. Hence $rank(Co_1 : 2A) = 3$. Darafsheh et. al in [6] proved that Co_1 is $(2Y, 3D, 11A)$-generated for $Y \in \{B, C\}$. Now by applying Corollary 1.2, we have $rank(Co_1 : 2Y) \leq 3$ for $Y \in \{B, C\}$, but we know that $rank(Co_1 : 2Y) > 2$ as we argue in the above case, hence $rank(Co_1 : 2Y) = 3$ where $Y \in \{B, C\}$. Therefore $rank(Co_1 : 2X) = 3$ where $X \in \{A, B, C\}$ □

Lemma 2.3. $rank(Co_1 : 3A) = 3$.

Proof. First we show that $rank(Co_1 : 3A) > 2$ by proving that Co_1 is not $(3A, 3A, tX)$-generated for any conjugacy class tX. If Co_1 is $(3A, 3A, tX)$-generated then $\frac{1}{3} + \frac{1}{3} + \frac{1}{t} < 1$ and it follows that $t \geq 4$. Set $K = \{4A, 5A, 6A\}$. Using $\mathbb{GAP}$ [17] we see that $\Delta_{Co_1}(3A, 3A, tX) = 0$ if $tX \notin K$ and for $tX \in K$ we have $\Delta_{Co_1}(3A, 3A, tX) < |C_{Co_1}(tX)|$. We get that

$$\Delta^*_{Co_1}(3A, 3A, tX) < \Delta_{Co_1}(3A, 3A, tX) < |C_{Co_1}(tX)|.$$

Using Lemma 1.4, we obtain that $\Delta^*_{Co_1}(3A, 3A, tX) = 0$ for all tX with $t \geq 4$ and therefore Co_1 is not $(3A, 3A, tX)$-generated and hence $rank(Co_1 : 3A) > 2$. Next we show that $rank(Co_1 : 3A) = 3$.

Consider the triple $(3A, 3A, 3A, 10E)$. From the maximal subgroups of Co_1, we see that the only maximal subgroups of Co_1 with order divisible by 10 and non-empty intersection with the conjugacy classes $3A$ and $10E$ are isomorphic to $H_1 = 2^{1+8}_+.O^+_8(2)$, $H_2 = 3^{1+4}.2U_4(2).2$, $H_3 = (A_5 \times J_2)$:2 and $H_4 = (D_{10} \times (A_5 \times A_5).2).2$. We compute $\Delta_{Co_1}(3A, 3A, 3A, 10E) = 600$ and $\Sigma_{H_1}(3A, 3A, 3A, 10E) = \Sigma_{H_2}(3A, 3A, 3A, 10E) = \Sigma_{H_3}(3A, 3A, 3A, 10E) = \Sigma_{H_4}(3A, 3A, 3A, 10E) = 0$. Thus no proper subgroup of Co_1 is $(3A, 3A, 3A, 10E)$-generated and we get

$$\Delta^*_{Co_1}(3A, 3A, 3A, 10E) = \Delta_{Co_1}(3A, 3A, 3A, 10E) = 600.$$

Hence Co_1 is $(3A, 3A, 3A, 10E)$-generated and the result follows. □

Lemma 2.4. $rank(Co_1 : tX) = 2$ *for* $tX \in \{3B, 4A, 4B, 4C, 4D, 5A, 6A\}$.

Proof. Set $T = \{3B, 4B, 4D, 5A, 6A\}$. Consider the triple $(tX, tX, 13A)$ for each $tX \in T$. The maximal subgroups of Co_1 containing elements of order 13 are, up to isomorphism, $H_1 \cong 3.Suz.2$ and $H_2 \cong (A_4 \times G_2(4))$:2. We see that a fixed element of order 13 in Co_1 is contained in precisely four copies of H_1 in Co_1 and in a unique conjugate copy of H_2 in Co_1. Now we calculate that for each $tX \in T$, we have $\Delta^*_{Co_1}(tX, tX, 13A) \geq \Delta_{Co_1}(tX, tX, 13A) - 4\Sigma_{H_1}(tX, tX, 13A) - \Sigma_{H_2}(tX, tX, 13A) > 0$. We conclude that Co_1 is $(tX, tX, 13A)$-generated for each $tX \in T$. Hence $rank(Co_1 : tX) = 2$ for each $tX \in T$.

Next for $tX = 4A$ consider the triple $(2C, 4A, 26A)$. Up to isomorphism, the only maximal subgroup of Co_1 that may contain (2C,4A,26A)-generated proper subgroup is isomorphic to $H_2 \cong (A_4 \times G_2(4))$:2. We calculate that $\Delta_{Co_1}(2C, 4A, 26A) = 91$ and $\Sigma_{H_2}(2C, 4A, 26A) = 39$. Now a fixed element of order 26 in Co_1 lies in a unique conjugate of H_2 in Co_1. Hence

H_2 contributes $1 \times 39 = 39$ to the number $\Delta_{Co_1}(2C, 4A, 26A)$. Our calculation gives $\Delta^*_{Co_1}(2C, 4A, 26A) \geq 91 - 39 > 0$ and therefore, Co_1 is $(2C, 4A, 26A)$-generated. Now applying Lemma 1.2, we get $rank(Co_1 : 4A) = 2$.

Finally for the rank of the conjugacy class $tX = 4C$ we consider the triple $(4C, 4C, 13A)$. The Co_1-class $4C$ fails to meet any copy of H_1 or H_2 in Co_1. Thus Co_1 contains no proper $(4C, 4C, 13A)$-subgroup. As $\Delta_{Co_1}(4C, 4C, 13A) = 7866268$ we conclude that Co_1 is $(4C, 4C, 13A)$-generated and $rank(Co_1 : 4C) = 2$. This completes the proof. □

Lemma 2.5. *If $n \geq 4$ and $nX \notin T = \{4A, 4B, 4C, 4D, 5A, 6A\}$ then $rank(Co_1 : nX) = 2$.*

Proof. Direct computation using $\mathbb{GAP}$ and results from Darafsheh, Arshafi and Moghani ([8]) together with information about the power maps of Co_1 we can show that Co_1 is $(2A, nX, mZ)$-generated for each conjugacy class $nX \notin T$ of Co_1 $(n \geq 4)$ with appropriate mZ. Now by Lemma 1.3, Co_1 is $(nX, nX, (mZ)^2)$-generated for all $nX \notin T$ $(n \geq 4)$. Hence $rank(Co_1 : nX) = 2$ for all $n \geq 4$ and for each conjugacy class $nX \notin T$ of Co_1. □

Remark 2.6. For example Co_1 is $(2A, 23A, 23B)$-generated. Hence Co_1 is $(23A, 23A, (23B)^2)$-generated, so that $rank(Co_1 : 23A) = 2$.

We now state the main result of the paper.

Theorem 2.7. *Let Co_1 be the Conway's largest sporadic simple group. Then*

(a) $rank(Co_1 : nX) = 3$ *if* $nX \in \{2A, 2B, 2C, 3A\}$.

(b) $rank(Co_1 : nX) = 2$ *if* $nX \notin \{1A, 2A, 2B, 2C, 3A\}$.

Proof. The proof follows from Lemmas 2.1, 2.2, …, and 2.5. □

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