On the ranks of HS and McL

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Abstract

If G is a finite group and X a conjugacy class of G, then we define $rank(G : X)$ to be the minimum number of elements of X generating G. In the present paper, we determine the ranks of the sporadic simple groups HS and McL. Most of the calculations were carried out using the computer algebra system $\mathbb{GAP}$ [13].

1 Introduction and Preliminaries

Let G be a finite group and $X \subseteq G$. We denote the minimum number of elements of X generating G by $rank(G : X)$. In the present paper we investigate $rank(G : X)$ where X is a conjugacy class of G and G is a sporadic simple group.

Moori in [9], [10] and [11] proved that $5 \leq rank(Fi_{22} : 2A) \leq 6$ and $rank(Fi_{22} : 2B) = rank(Fi_{22} : 2C) = 3$ where $2A$, $2B$ and $2C$ are the conjugacy classes of involutions of the smallest Fischer group Fi_{22} as represented in the $\mathbb{ATLAS}$ [1]. Hall and Soicher in [6] proved that $rank(Fi_{22} : 2A) = 6$. Moori in [12] determined the ranks of the Janko group J_1, J_2 and J_3.

In the present paper, we determine the ranks of the two sporadic simple groups, namely Higman-Sims group HS and McLaughlin group McL. For basic properties of HS and McL, character tables of these groups and their maximal subgroups we use $\mathbb{ATLAS}$ [1] and $\mathbb{GAP}$ [13]. For detailed information about the computational

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techniques used in this paper the reader is encouraged to consult [5], [11] and [12].

We now develop the terminology and notation that will be used in the subsequent sections. Throughout this paper we use the same notation as in [5] and [11]. In particular, for a finite group G with $C_1, C_2, \ldots, C_k$ conjugacy classes of its elements and g_k a fixed representative of C_k, we denote $\Delta_G(C_1, C_2, \ldots, C_k)$ the number of distinct tuples $(g_1, g_2, \ldots, g_{k-1})$ with $g_i \in C_i$ such that $g_1 g_2 \ldots g_{k-1} = g_k$. It is well known that $\Delta_G(C_1, C_2, \ldots, C_k)$ is structure constant for the conjugacy classes $C_1, C_2, \ldots, C_k$ and can be easily computed from the character table of G (see [7], p.45) by the following formula

$$\Delta_G(C_1, C_2, \ldots, C_k) = \frac{|C_1||C_2| \ldots |C_{k-1}|}{|G|} \times \sum_{i=1}^{m} \frac{\chi_i(g_1)\chi_i(g_2) \ldots \chi_i(g_{k-1})\overline{\chi_i(g_k)}}{[\chi_i(1_G)]^{k-2}}$$

where $\chi_1, \chi_2, \ldots, \chi_m$ are the irreducible complex characters of G. Further let $\Delta^*_G(C_1, C_2, \ldots, C_k)$ denote the number of distinct tuples $(g_1, g_2, \ldots, g_{k-1})$ with $g_i \in C_i$ and $g_1 g_2 \ldots g_{k-1} = g_k$ such that $G =< g_1, g_2, \ldots, g_{k-1} >$. If $\Delta^*_G(C_1, C_2, \ldots, C_k) > 0$, then we say that G is $(C_1, C_2, \ldots, C_k)$-generated. If H is a subgroup of G containing g_k and B is a conjugacy class of H such that $g_k \in B$, then $\Sigma_H(C_1, C_2, \ldots, C_{k-1}, B)$ denotes the number of distinct tuples $(g_1, g_2, \ldots, g_{k-1})$ such that $g_i \in C_i$ and $g_1 g_2 \ldots g_{k-1} = g_k$ and $\langle g_1, g_2, \ldots, g_{k-1} \rangle \leq H$.

For the description of the conjugacy classes, the character tables, permutation characters and information on the maximal subgroups readers are referred to $\mathbb{ATLAS}$ [1]. A general conjugacy class of elements of order n in G is denoted by nX. For example $2A$ represents the first conjugacy class of involutions in a group G. We will use the maximal subgroups and the permutation characters of HS and McL on the conjugates (right cosets) of the maximal subgroups listed in the $\mathbb{ATLAS}$ [1] extensively.

The following results will be crucial in determining the ranks of a finite group G.

Lemma 1 *(Moori [12]) Let G be a finite simple group such that G is (lX, mY, nZ)-generated. Then G is $(\underbrace{lX, lX, \ldots, lX}_{m-times}, (nZ)^m)$-generated.*

Corollary 2 *Let G be a finite simple group such that G is (lX, mY, nZ)-generated, then $rank(G : lX) \leq m$.*

Proof: The proof follows immediately from Lemma 1.

Lemma 3 *(Conder et al. [2]) Let G be a simple $(2X, mY, nZ)$-generated group. Then G is $(mY, mY, (nZ)^2)$-generated.*

The following lemma gives useful criterion for establishing non-generation.

Lemma 4 *([17]) Let G be a finite centerless group and suppose lX, mY, nZ are G-conjugacy classes for which $\Delta^*(G) = \Delta^*_G(lX, mY, nZ) < |C_G(nZ)|$. Then $\Delta^*(G) = 0$ and therefore G is not (lX, mY, nZ)-generated.* □

2 Ranks of HS

The Higman-Sims group HS is a sporadic simple group of order $2^9.3^2.5^3.7.11$ with 12 classes of maximal subgroups. HS has 24 conjugacy classes of its elements. It has two conjugacy classes of involutions namely $2A$ and $2B$. The group HS acts primitively on a set Ω of 100 points. The point stabilizer of this action is isomorphic to the Mathieu group M_{22} and the orbits have length 1, 22 and 77. The permutation character of HS on the conjugates of M_{22} is given by $\chi_{M_{22}} = \underline{1a} + \underline{22a} + \underline{77a}$. For basic properties of HS and computational techniques, the reader is encouraged to consult [5], [9] and [10].

We now compute rank of each conjugacy class of HS.

It is well known that every sporadic simple group can be generated by three involutions (see [3]). In the following lemma we prove that HS can be generated by three involutions $a, b, c \in 2X$, where $X \in \{A, B\}$ such that $abc \in 11A$

Lemma 5 *rank*$(HS : 2X) = 3$ *where* $X \in \{A, B\}$.

Proof. We know that HS is $(2B, 3A, 11A)$-generated by Ganief and Moori [5] and Wolder [16]. By applying Corollary 2, we have *rank*$(HS : 2B) \leq 3$. But *rank*$(HS : 2B) = 2$ is not possible, because if $\langle x, y\rangle = HS$ for some $x, y \in 2B$ then $HS \cong D_{2n}$ with $o(xy) = n$. Hence *rank*$(HS : 2B) = 3$.

For the rank of the conjugacy class $2A$, we first show that HS is $(2A, 2A, 2A, 11A)$-generated. We compute the structure constant $\Delta_{HS}(2A, 2A, 2A, 11A) = 3872$. If z is a fixed element of order 11 in HS, then there are 3872 distinct triples $(x, x^{'}, x^{''})$ with $\{x, x^{'}, x^{''}\} \subset 2A$ such that $xx^{'}x^{''} = z$. We observe that the only maximal subgroups of HS which have order divisible by 11, up to isomorphism, are M_{11} (two non-conjugate copies) and M_{22}. Clearly then, any proper $(2A, 2A, 2A, 11A)$-subgroup of HS must lie in one of M_{11} or M_{22}. In M_{11}, the $2A$-class, say T, is the only class which fuses to $2A$-class of HS and we obtain that $\Sigma_{M_{11}}(2A, 2A, 2A, 11A) = \Delta_{M_{11}}(2A, 2A, 2A, 11A) = 605$. Since z is contained in precisely one conjugate of each M_{11} in HS. Thus the the total contribution from subgroups of HS isomorphic to M_{11} to the distinct triples $(x, x^{'}, x^{''})$ with $\{s, s^{'}, s^{''}\} \subset T$ and $xx^{'}x^{''} = z$ is equal to 605×2.

Similarly, we compute $\Sigma_{M_{22}}(2A, 2A, 2A, 11A) = 2420$. Since z is contain in precisely one conjugate of M_{22} in HS, the total contribution from the subgroups of HS

isomorphic to M_{22} to the distinct triples (x, x', x'') in M_{22} with $xx'x'' = z$ is equal to 2420.

Thus we have

$$\begin{aligned}\Delta^*_{HS}(2A, 2A, 2A, 11A) &\geq \Delta_{HS}(2A, 2A, 2A, 11A) - [2 \times \Sigma_{M_{11}}(2A, 2A, 2A, 11A) \\ &\quad +\Sigma_{M_{22}}(2A, 2A, 2A, 11A)] \\ &= 3872 - [2 \times 605 + 2420] > 0.\end{aligned}$$

Hence HS is $(2A, 2A, 2A, 11A)$-generated and therefore we have $rank(HS : 2A) \leq 3$. Since $rank(HS : 2A) > 2$, the result follows.

Remark 1 *The converse of Lemma 1 is not true in general since HS is not $(2A, 3A, tZ)$-generated group for any tZ.*

Table I
Structure Constants of HS

tX	$3A$	$4A$	$4B$	$4C$	$5A$	$5B$	$5C$	$6A$	$6B$
$\Delta_{HS}(4A, 4A, tX)$	75	0	4	32	0	60	10	0	0
$\lvert C_{HS}(tX)\rvert$	360	3840	256	64	500	300	25	36	24
tX	$7A$	$8ABC$	$10A$	$10B$	$11AB$	$12A$	$15A$	$20AB$	
$\Delta_{HS}(4A, 4A, tX)$	7	0	13	0	0	0	15	20	
$\lvert C_{HS}(tX)\rvert$	7	16	20	20	11	12	15	20	

Lemma 6 $rank(HS : 4A) = 3$.

Proof. First we show that HS can not be $(4A, 4A, tX)$-generated for any tX. If HS is $(4A, 4A, tX)$-generated then $\frac{1}{4} + \frac{1}{4} + \frac{1}{t} < 1$ and it follows that $t \geq 3$. Set $K = \{3A, 4A, 4B, 4C, 5A, 5B, 5C, 6A, 6B, 8A, 8B, 8C, 10A, 10B, 11A, 11B, 12A, 15A, 20A, 20B\}$. If $tX \in K$ then from Table I, we see that

$$\Delta^*_{HS}(4A, 4A, tX) \leq \Delta_{HS}(4A, 4A, tX) < |C_{HS}(tX)|.$$

Now by using Lemma 4, we obtain that $\Delta^*_{HS}(4A, 4A, tX) = 0$ and hence HS is not $(4A, 4A, tX)$-generated for every $tX \in K$.

For the triple $(4A, 4A, 7A)$, we have $\Delta_{HS}(4A, 4A, 7A) = 7 = |C_{HS}(7A)|$. To show that HS is not $(4A, 4A, 7A)$-generated, we construct the HS using its "standard generators" given in [14] and also in [15]. The group HS has 20-dimensional irreducible representation over $GF(2)$. Using $\mathbb{GAP}$ we generate $HS = \langle a, b\rangle$, where a and b are 20×20 matrices over $GF(2)$ with orders 2 and 5 respectively. Let $x = ((ab)^{-7}(abababababbab^3)^3(ab)^6)^3$ and $z = (a^6(ab)^2(ab^2)^{27}(abab^2ab^2)^{50}(ab^5))^2$. Using $\mathbb{GAP}$ we see that $a \in 2A$, $b \in 5A$ and $ab \in 11A$. Also $x \in 4A$ and $z \in 7A$. Now if $y = (x^2z)^3$ then $y \in 4A$ and $xy \in 7A$. Let $P = \langle x, y\rangle$ then $P < HS$ and $P \cong S_7$. We calculate that $\Sigma_P(4A, 4A, 7A) = 7$. By investigating the maximal subgroups of P and their fusions into P and HS, we find that no maximal subgroup of P is $(4A, 4A, 7A)$-generated and hence no proper subgroup of P is $(4A, 4A, 7A)$-generated. Thus $\Delta^*_{HS}(4A, 4A, 7A) = 0$ and non-generation by this triple follows. Hence HS is not $(4A, 4A, tX)$-generated for any t and we conclude that $rank(HS : 3A) > 2$.

Next we show that HS is $(4A, 4A, 4A, 10A)$-generated. Using character table of HS, we compute the structure constant $\Delta_{HS}(4A, 4A, 4A, 10A) = 22800$. The maximal subgroups of HS with elements of order 10 and nontrivial intersection with classes $4A$ and $10A$ are, up to isomorphism, $U_3(5){:}2$ (two non-conjugate copies), $4{\cdot}2^4{:}S_5$ and $5{:}4 \times A_5$. An easy computation reveals that

$$\Sigma_{U_3(5):2}(4A, 4A, 4A, 10A) = \Sigma_{4\cdot 2^4:S_5}(4A, 4A, 4A, 10A) = \Sigma_{5:4\times A_5}(4A, 4A, 4A, 10A) = 0.$$

It follows that $\Delta^*_{HS}(4A, 4A, 4A, 10A) = \Delta_{HS}(4A, 4A, 4A, 10A) = 22800$. Thus HS has no proper $(4A, 4A, 4A, 10A)$-generated subgroup, so is itself $(4A, 4A, 4A, 10A)$-generated. Since $rank(HS : 4A) > 2$, the result follows.

Theorem 7 *If $nX \notin \{1A, 2A, 2B, 4A\}$ then $rank(HS : nX) = 2$.*

Proof. First we treat the case when $nX = 3A$. Since HS is $(2B, 3A, 11A)$-generated (see Wolder [16]). By Lemma 3, HS is $(3A, 3A, (11Z)^2)$-generated. Hence we have $rank(HS : 3A) = 2$. Now for the conjugacy class $nX = 4B$, consider the triple $(4B, 4B, 10A)$. Here the structure constant $\Delta_{HS}(4B, 4B, 10A) = 375$. A quick examination of the maximal subgroup structure of HS reveals that any $(4B, 4B, 10A)$-generated subgroup must be contained in $4{\cdot}2^4{:}S_5$. Since $\Sigma_{4\cdot 2^4:S_5}(4B, 4B, 10A) = 15$, we have $\Delta^*_{HS}(4B, 4B, 10A) \geq 375 - 15 > 0$. Hence HS is $(4B, 4B, 10A)$-generated and so $rank(HS : 4B) = 2$.

Direct computation using $\mathbb{GAP}$ and results from Ganief and Moori [5] show that $HS = \langle a, b\rangle$ where $a \in 2A$, and $b \in nX$ with $nX \in \{4C, 5A, 5B, 5C, 6A, 6B, 7A, 11A, 11B\}$. Since $(8B)^2 = 4C = (8C)^2$, $(10A)^2 = 5A$, $(10B)^2 = 5B$ and $(20A)^2 = 10A = (20B)^2$, $(12A)^2 = 6B$ and $(15A)^3 = 5B$ we have $HS = \langle a, c\rangle$ where $c \in \{8B, 8C, 10A, 10B, 12A, 15A, 20A, 20B\}$.

Therefore HS is $(2A, nX, mY)$-generated where $nX \in \{4C, 5A, 5B, 5C, 6A, 6B, 7A, 8A, 8B, 8C, 10A, 10B, 11A, 11B, 12A, 15A, 20A, 20B\}$ with appropriate mY. Hence $rank(HS : nX) = 2$ where $nX \notin \{1A, 2A, 2B, 4A\}$. □

3 Ranks of McL

The sporadic simple group of McLaughlin McL has order $2^7.3^6.5^3.7.11$ with 24 conjugacy classes of its elements. It has only one class of involutions, namely $2A$. We adopt the same notation as in the previous section. For information regarding the maximal subgroups and other background material about McL, the interested reader is referred to [1] and [4].

Before investigating the ranks of McL we show that McL can be generated by the three involutions.

Lemma 8 *The group McL is $(2A, 2A, 2A, 11A)$-generated.*

Proof. Observe that the only maximal subgroups of McL which have order divisible by 11 and non-empty intersection with the classes $2A$ and $11A$ are isomorphic to M_{11} and M_{22} (two non-conjugate copies). We calculate $\Delta_{McL}(2A, 2A, 2A, 11A) = 9317$, $\Sigma_{M_{22}}(2A, 2A, 2A, 11A) = 2420$, $\Sigma_{M_{11}}(2A, 2A, 2A, 11A) = 605$. Further, a fixed element of order 11 is contained in two conjugates of a M_{22} subgroup and a unique sonjugate of a M_{11} subgroup. Thus

$$\begin{aligned} \Delta^*_{McL}(2A, 2A, 2A, 11A) &\geq \Delta_{McL}(2A, 2A, 2A, 11A) - 2 \times \Sigma_{M_{22}}(2A, 2A, 2A, 11A) \\ &\quad -\Sigma_{M_{11}}(2A, 2A, 2A, 11A), \\ &= 9317 - 2 \times 2420 - 605 > 0. \end{aligned}$$

Hence McL is $(2A, 2A, 2A, 11A)$-generated. □

Lemma 9 *$rank(McL : 2A) = 3$.*

Proof. In the previous Lemma, we showed that McL can be generated by three involutions $x, y, z \in 2A$ such that $xyz \in 11A$. Therefore $rank(McL : 2A) \leq 3$. Since $rank(McL : 2A) = 2$ is not possible, the result follows. □

Lemma 10 *$rank(McL : 3A) = 3$*

Proof. Since McL is $(3A, 5X, 11Y)$-generated where $X, Y \in \{A, B\}$ (see Ganief and Moori [5]), we have $2 \leq rank(McL : 3A) \leq 5$. If the group McL is $(3A, 3A, tX)$-generated then $\frac{1}{3} + \frac{1}{3} + \frac{1}{t} < 1$ and it follows that $t \geq 4$. It is evident from Table II that $\Delta_{McL}(3A, 3A, tX) < |C_{McL}(3A, 3A, tX)|$ for all $t \geq 4$. Therefore by Lemma 4, $\Delta^*_{McL}(3A, 3A, tX) = 0$ and we conclude that McL is not $(3A, 3A, tX)$-generated for any tX. Hence $rank(McL : 3A) > 2$.

Now consider the case $(3A, 3A, 3A, 10A)$. The maximal subgroups of McL with non-empty intersection with the classes $3A$ and $10A$ are, up to isomorphisms, $H \cong 3^{1+4}{:}2S_5$, $L \cong 2.A_8$ and $U \cong 5^{1+2}{:}3{:}8$. Also, $\Delta_{McL}(3A, 3A, 3A, 10A) = 27650$, $\Sigma_H(3A, 3A, 3A, 10A) = 50$, $\Sigma_L(3A, 3A, 3A, 10A) = 50$ and $\Sigma_U(3A, 3A, 3A, 10A) = 0$. Since a fixed element of order 10 is contained in a unique conjugate of H, L and U subgroups of McL, respectively. We have

$$\Delta^*_{McL}(3A, 3A, 3A, 10A) \geq 27650 - [50 + 50 + 0] = 27550 > 0.$$

So, McL is $(3A, 3A, 3A, 10A)$-generated. Now the result follows from the relation that $rank(McL : 3A) > 2$. □

Table II
Structure Constants of McL

tX	$4A$	$5A$	$5B$	$6A$	$6B$	$7AB$	$8A$	$9AB$
$\Delta_{McL}(3A,3A,tX)$	4	0	10	15	0	0	0	0
$\lvert C_{McL}(tX)\rvert$	96	750	25	360	36	14	8	27
tX	$10A$	$11AB$	$12A$	$14AB$	$15AB$	$30AB$		
$\Delta_{McL}(3A,3A,tX)$	5	0	0	0	0	0		
$\lvert C_{McL}(tX)\rvert$	30	11	12	14	30	30		

Theorem 11 *If $nX \notin \{1A, 2A, 3A\}$ then rank$(McL : nX) = 2$.*

Proof. For $nX = 3B$ we show that McL is $(3B, 3B, 10A)$-generated. The maximal subgroups of McL which have non-empty intersection with the classes $3B$ and $10A$ are $H_1 \cong U_3(5)$, $H_2 \cong 3^{1+4}{:}2S_5$ and $H_3 \cong 2.A_8$. Any easy computation reveals that $\Delta_{McL}(3B, 3B, 10A) = 1375$, $\Sigma_{H_1}(3B, 3B, 10A) = 125$, $\Sigma_{H_2}(3B, 3B, 10A) = 10$ and $\Sigma_{H_3}(3B, 3B, 10A) = 55$. Thus we have

$$\begin{aligned}\Delta^*_{McL}(3B,3B,10A) &\geq \Delta_{McL}(3B,3B,10A) - [\Sigma_{H_1}(3B,3B,10A) \\ &\quad +\Sigma_{H_2}(3B,3B,10A) + \Sigma_{H_3}(3B,3B,10A)] \\ &= 1375 - [125 + 10 + 55] > 0.\end{aligned}$$

Hence McL is $(3B, 3B, 10A)$-generated and therefore we obtain that *rank*$(McL : 3B) = 2$.

Direct computation using $\mathbb{GAP}$ and from the results of Ganief and Moori ([5]) together with information about the power maps, we can be show that McL is $(2A, nX, mZ)$-generated for all $nX \notin \{1A, 2A, 3A, 3B\}$ with appropriate mZ. Now by Lemma 3, McL is $(nX, nX, (mZ)^2)$-generated for all $nX \notin \{1A, 2A, 3A, 3B\}$. Hence *rank*$(McL : nX) = 2$ where $nX \notin \{1A, 2A, 3A\}$. □

Remark 2 *For example McL is $(2A, 7A, 11X)$-generated where $X \in \{A, B\}$. Hence McL is $(7A, 7A, (11X)^2)$-generated and so rank$(McL : 7A) = 2$.*

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