On the ranks of the *Conway* groups Co_2 and Co_3*

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Abstract

Let G be a finite group and X a conjugacy class of G. We denote rank(G:X) to be the minimum number of elements of X generating G. In the present article, we determine the ranks of the *Conway* groups Co_2 and Co_3. Computations were carried with the aid of computer algebra system $\mathbb{GAP}$ [18].

1 Introduction and Preliminaries

There has recently been some interest in generation of simple groups by their conjugate involutions. It is well known that sporadic simple groups are generated by three conjugate involutions (see [5]). If a group $G = \langle a, b\rangle$ is perfect and $a^2 = b^3 = 1$ then clearly G is generated by three conjugate involutions a, a^b and a^{b^2} (see [6]). Moori [15] proved that the Fischer group Fi_{22} can be generated by three conjugate involutions. The work of Liebeck and Shalev [12] shows that all but finitely many classical groups can be generated by three involutions. However, the problem of finding simple classical groups which can be generated by three conjugate involutions is still very much open. The generation of a simple group by its conjugate elements in this context is of some interest. Therefore, we concentrate on the generation of simple groups by their conjugate elements and investigate two sporadic simple groups.

Suppose that G is a finite group and $X \subseteq G$. We denote the rank of X in G by rank(G:X), the minimum number of elements of X generating G. This paper focuses

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on the determination of rank(G:X) where X is a conjugacy class of G and G is a sporadic simple group.

Moori in [13], [14] and [15] proved that rank(Fi_{22}:$2A$) $\in \{5, 6\}$ and rank(Fi_{22}:$2B$) = rank(Fi_{22}:$2C$) = 3 where $2A, 2B$ and $2C$ are the conjugacy classes of involutions of the smallest Fischer group Fi_{22} as presented in the $\mathbb{ATLAS}$ [3]. The work of Hall and Soicher [9] show that rank(Fi_{22}:$2A$) = 6. Moori [16] determined the ranks of the *Janko* groups J_1, J_2 and J_3. More recently, in [1], the authors investigated the ranks of *Higman-Sims* group HS and *McLaughlin* group McL. In the present article we continue our study on the ranks of sporadic simple groups and determine the ranks of the *Conway's* sporadic simple groups Co_2 and Co_3.

For basic properties of Co_2 and Co_3, character tables of these groups and their maximal subgroups etc. we use $\mathbb{ATLAS}$ [3] and [18]. For detailed information about the computational techniques used in this paper the reader is encouraged to consult [1], [8], [15], and [16].

Next we discuss some background material and introduce the notation. We adopt the same notation as in the above mentioned papers. In particular, if G is a finite group, $C_1, C_2, \cdots, C_k$ are the conjugacy classes of its elements and g_k is a fixed representative of C_k, then $\Delta_G(C_1, C_2, \cdots, C_k)$ denotes the number of distinct tuples $(g_1, g_2, \cdots, g_{k-1}) \in (C_1 \times C_2 \times \cdots \times C_{k-1})$ such that $g_1 g_2 \cdots g_{k-1} = g_k$. It is well known that $\Delta_G(C_1, C_2, \cdots, C_k)$ is the structure constant of G for the conjugacy classes $C_1, C_2, \cdots, C_k$ and can be computed from the character table of G (see [11], p.45) by the following formula

$$\Delta_G(C_1, C_2, \cdots, C_k) = \frac{|C_1||C_2|\cdots|C_{k-1}|}{|G|} \times \sum_{i=1}^{m} \frac{\chi_i(g_1)\chi_i(g_2)\cdots\chi_i(g_{k-1})\overline{\chi_i(g_k)}}{[\chi_i(1_G)]^{k-2}}$$

where $\chi_1, \chi_2, \cdots, \chi_m$ are the irreducible complex characters of G. Also, $\Delta^*_G(C_1, C_2, \cdots, C_k)$ denotes the number of distinct tuples $(g_1, g_2, \cdots, g_{k-1}) \in (C_1 \times C_2 \times \cdots \times C_{k-1})$ such that $g_1 g_2 \cdots g_{k-1} = g_k$ and $G = \langle g_1, g_2, \cdots, g_{k-1} \rangle$. If $\Delta^*_G(C_1, C_2, \cdots, C_k) > 0$, then we say that G is $(C_1, C_2, \cdots, C_k)$-generated. If H any subgroup of G containing the fixed element $g_k \in C_k$, then $\Sigma_H(C_1, C_2, \cdots, C_{k-1}, C_k)$ denotes the number of distinct tuples $(g_1, g_2, \cdots, g_{k-1}) \in (C_1 \times C_2 \times \cdots \times C_{k-1})$ such that $g_1 g_2 \cdots g_{k-1} = g_k$ and $\langle g_1, g_2, \cdots, g_{k-1} \rangle \leq H$ where $\Sigma_H(C_1, C_2, \cdots, C_k)$ is obtained by summing the structure constants $\Delta_H(c_1, c_2, \cdots, c_k)$ of H over all H-conjugacy classes $c_1, c_2, \cdots, c_{k-1}$ satisfying $c_i \subseteq H \cap C_i$ for $1 \leq i \leq k-1$.

The $\mathbb{ATLAS}$ serves as a valuable source of information and we use the Atlas notation for conjugacy classes, maximal subgroups, character tables, permutation characters, etc. A general conjugacy class of elements of order n in G is denoted by nX. For examples, $2A$ represents the first conjugacy class of involutions in a group G. We will use the maximal subgroups and the permutations characters of Co_2 and Co_3 on the conjugates (right cosets) of the maximal subgroups listed in the $\mathbb{ATLAS}$ [3] extensively.

The following results will be crucial in determining the ranks of finite groups.

Lemma 1 *(Moori [16]) Let G be a finite simple group such that G is (lX, mY, nZ)-generated. Then G is $(\underbrace{lX, lX, \cdots, lX}_{m-times}, (nZ)^m)$-generated.*

Corollary 2 *Let G be a finite simple group such that G is (lX, mY, nZ)-generated, then* $\text{rank}(G : lX) \leq m$.

Proof: Immediately follows from Lemma 1. □

Lemma 3 *(Conder et al. [4]) Let G be a simple $(2X, mY, nZ)$-generated group. Then G is $(mY, mY, (nZ)^2)$-generated.*

2 Ranks of Co_2

The Conway group Co_2 is a sporadic simple group of order $2^{18}.3^6.5^3.7.11.23$ with 11 conjugacy classes of maximal subgroups. It has 60 conjugacy classes of its elements including three conjugacy classes of involutions, namely $2A, 2B$ and $2C$. The group Co_2 acts primitively on a set Ω of 2300 points. The point stabilizer of this action is isomorphic to $U_6(2){:}2$ and the orbits have length 1, 891 and 1408. The permutation character of Co_2 on the cosets of $U_6(2){:}2$ is given by $\chi_{U_6(2):2} = \underline{1a} + \underline{275a} + \underline{2024a}$ For basic properties of Co_2 and computational techniques, the reader is encouraged to consult [1], [2], [8] and [19].

We now compute the rank of each conjugacy class of Co_2.

It is well known that every sporadic simple group can be generated by three involutions (see [6]). In the following lemmas we prove that Co_2 can be generated by three involutions $a, b, c \in 2X$ where $X \in \{A, B\}$.

Lemma 4 *Co_2 is $(2B, 2B, 2B, 23A)$-generated.*

Proof: Simple computation show that the structure constant $\triangle_{Co_2}(2B, 2B, 2B, 23A) =$ 12696. If z is a fixed element of order 23 in Co_2 then there are 12696 distinct triples (α, β, γ) such $\{\alpha, \beta, \gamma\} \subset 2A$ and $\alpha\beta\gamma = z$. We observe that the only maximal subgroup of Co_2 which has order divisible by 23 is M_{23} and z is contained in a unique conjugate of M_{23}. Hence, any proper $(2B, 2B, 2B, 23A)$-subgroup of Co_2 must be in M_{23}. Furthermore, the $2A$-class is the only class which fuses to $2B$-class of Co_2 in M_{23}. It then follows that $\Sigma_{M_{23}}(2B, 2B, 2B, 23A) = 3174$. Thus the total contribution from M_{23} to the distinct triples (α, β, γ) with $\{\alpha, \beta, \gamma\} \subset 2A$ and $\alpha\beta\gamma = z$ is equal to 3174. Thus we have

$$\begin{aligned}\Delta^*_{Co_2}(2B, 2B, 2B, 23A) &\geq \Delta_{Co_2}(2B, 2B, 2B, 23A) - \Sigma_{M_{23}}(2B, 2B, 2B, 23A)\\ &= 12696 - 3174 > 0.\end{aligned}$$

Hence Co_2 is $(2B, 2B, 2B, 23A)$-generated. □

Lemma 5 rank$(Co_2 : 2X) = 3$ *where* $X \in \{B, C\}$.

Proof: Let $X \in \{B, C\}$. Ganief and Moori have shown in [8] that Co_2 is $(2C, 3A, 23A)$-generated. Thus, from the previous lemma and the above stated result from [8] together with application of Corollary 2 imply that $rank(Co_2 : 2X) \leq 3$. But the case $rank(Co_2 : 2X) = 2$ is not possible since if there are $x, y \in 2X$ such that $Co_2 =< x, y >$, then $Co_2 \cong D_{2n}$ where $n = o(xy)$. This concludes that $rank(Co_2 : 2X) = 3$ whenever $X \in \{B, C\}$. □

Lemma 6 *The group Co_2 is not $(2A, 2A, 2A, tX)$-generated for any conjugacy class tX in Co_2.*

Proof: The group Co_2 acts on a 275-dimensional irreducible complex module V. Let $d_{nX} = dim(V/C_V(nX))$, the co-dimension of the fix space (in V) of a representative in nX. Using the character table of Co_2 we list in Table I, the values of d_{nX}, for the conjugacy classe nX.

TABLE I
The co-dimensions $d_{nX} = dim(V/C_V(nX))$

d_{2A}	d_{2B}	d_{2C}	d_{4B}	d_{4C}	d_{4D}	d_{4E}	d_{4F}	d_{4G}
112	120	132	190	194	196	194	196	204

Set $T = \{2A, 2B, 2C, 4B, 4C, 4D, 4E, 4F, 4G\}$. If the group Co_2 is $(2A, 2A, 2A, tX)$-generated, then by Scott's theorem (see [4] and [17]) we must have

$$d_{2A} + d_{2A} + d_{2A} + d_{tX} \geq 2 \times 275.$$

However, it is clear from Table I that $3 \times d_{2A} + d_{tX} < 550$ for each $tX \in T$ and therefore Co_2 is not $(2A, 2A, 2A, tX)$-generated, for each $tX \in T$.

Next suppose that $tX \notin T$ then computing the structure constants, we see that

$$\Delta_{Co_2}(2A, 2A, 2A, tX) < |C_{Co_2}(tX)|$$

for each $tX \notin T$ except for $tX = 12H$. Now an application of Lemma 3.3 in [23] shows that Co_2 is not $(2A, 2A, 2A, tX)$-generated for any $tX \notin T$ and $tX \neq 12H$.

Finally we consider the only remaining case $(2A, 2A, 2A, 12H)$. For this case we have $\Delta_{Co_2}(2A, 2A, 2A, 12H) = 72$ and $|C_{Co_2}(z)| = 48$, $z \in 12H$. In order to show that Co_2 is not $(2A, 2A, 2A, 12H)$-generated we construct the group Co_2 by using its "standard generators" given in [21] and also in [20]. The group Co_2 has a

22-dimensional irreducible representation over $GF(2)$. Using this representation we generate $Co_2 = \langle a, b\rangle$, where a and b are 22×22 matrices over $GF(2)$ with orders 2 and 5 respectively such that ab has order 28. Using $\mathbb{GAP}$, we see that $a \in 2A$, $b \in 5A$ and $ab \in 28A$. We produce $x = b^{-3}ab^3$, $y = ((babab)^2 b^{11})^7$ and $z = axy$ such that $x, y \in 2A$ and $z \in 12H$. Let $H = \langle a, x, y\rangle$ then $H < Co_2$ with $|H| = 1152$. We compute that $\Sigma_H(2A, 2A, 2A, 12H) = 72$ and consequently $\Delta^*_{Co_2}(2A, 2A, 2A, 12H) = 0$. Hence Co_2 is not $(2A, 2A, 2A, 12H)$-generated. This completes the proof. □

Lemma 7 rank$(Co_2 : 2A) = 4$.

Proof: Direct computation using $\mathbb{GAP}$ [18] and the results of [8] we see that $Co_2 = \langle x, y\rangle$ such that $x \in 2A$, $y \in 4G$ with $xy \in 23A$. Therefore, Co_2 is $(2A, 4G, 23A)$-generated. Now, it follows by Corollary 2 that rank$(Co_2 : 2A) \leq 4$. Since rank$(Co_2 : 2A) > 3$ by the above lemma, the result follows. □

Lemma 8 *The group Co_2 is $(2C, tX, 23A)$-generated where $tX \in \{3A, 3B, 4C, 4E\}$.*

Proof: We observe that the only maximal subgroup of Co_2 which has order divisible by 23 is M_{23} and $M_{23} \cap 2C = \emptyset$. Hence

$$\begin{aligned}
\Delta^*_{Co_2}(2C, 3A, 23A) &= \Delta_{Co_2}(2C, 3A, 23A) = 69 > 0,\\
\Delta^*_{Co_2}(2C, 3B, 23A) &= \Delta_{Co_2}(2C, 3B, 23A) = 69 > 0,\\
\Delta^*_{Co_2}(2C, 4C, 23A) &= \Delta_{Co_2}(2C, 4C, 23A) = 345 > 0,\\
\Delta^*_{Co_2}(2C, 4E, 23A) &= \Delta_{Co_2}(2C, 4E, 23A) = 3128 > 0.
\end{aligned}$$

Thus Co_2 is $(2C, tX, 23A)$-generated for any $tX \in \{3A, 3B, 4C, 4E\}$. □

Corollary 9 *Let $tX \in \{3A, 3B, 4C, 4E\}$. Then* rank$(Co_2 : tX) = 2$.

Proof: From the previous lemma we know that Co_2 is $(2C, tX, 23A)$-generated for any $tX \in \{3A, 3B, 4C, 4E\}$. Now result follows applying Lemma 3. □

Lemma 10 *The group Co_2 is $(4X, 4X, 10A)$-generated where $X \in \{A, B\}$.*

Proof: The structure constants $\Delta_{Co_2}(4A, 4A, 10A) = 125$. The only maximal subgroups of Co_2 which have non-empty intersection with the classes $4A$ and $10A$ are, up to isomorphism, $K_1 \cong (2^{1+6}_+ \times 2^4).A_8$ and $K_2 \cong 3^{1+4}_+ : 2^{1+4}_-.S_5$. Direct computation on $\mathbb{GAP}$ shows that $\Sigma_{K_1}(4A, 4A, 10A) = 5$ and $\Sigma_{K_2}(4A, 4A, 10A) = 5$. It then follows that $\Delta^*_{Co_2}(4A, 4A, 10A) \geq 125 - 1(5) - 4(5) > 0$. Hence Co_2 is $(4A, 4A, 10A)$-generated.

Similarly, the structure constant $\Delta_{Co_2}(4B, 4B, 23A) = 989$, and the only maximal subgroup of Co_2 which has an order divisible by 23 is M_{23} but $M_{23} \cap 4B = \emptyset$. This implies that

$$\Delta^*_{Co_2}(4B, 4B, 23A) = \Delta_{Co_2}(4B, 4B, 23A) = 989 > 0$$

and Co_2 is $(4B, 4B, 23A)$-generated. □

Corollary 11 *Let* $tX \in \{4A, 4B\}$. *Then* $\text{rank}(Co_2 : tX) = 2$.

Proof: This is clear from the previous lemma. □

Lemma 12 *If* $tX \in \{4D, 4F, 6C, 6D\}$ *then* $rank(Co_2 : tX) = 2$.

Proof: Again, the only maximal subgroup of Co_2 which has an element of order 23 is M_{23} but $M_{23} \cap tX = \emptyset$ for every $tX \in \{4D, 4F, 6C, 6D\}$. We obtained

$$\Delta^*_{Co_2}(2B, 4D, 23A) = \Delta_{Co_2}(2B, 4D, 23A) = 23 > 0,$$

$$\Delta^*_{Co_2}(2B, 4F, 23A) = \Delta_{Co_2}(2B, 4F, 23A) = 92 > 0,$$

$$\Delta^*_{Co_2}(2B, 6C, 23A) = \Delta_{Co_2}(2B, 6C, 23A) = 92 > 0,$$

$$\Delta^*_{Co_2}(2B, 6D, 23A) = \Delta_{Co_2}(2B, 6D, 23A) = 115 > 0.$$

Hence Co_2 is $(2B, tX, 23A)$-generated where $tX \in \{4D, 4F, 6C, 6D\}$ and we get that $rank(Co_2 : tX) = 2$ where $tX \in \{4D, 4F, 6C, 6D\}$. □

Theorem 13 *If* $nX \notin \{1A, 2A, 2B, 2C\}$, *then* $\text{rank}(Co_2 : nX) = 2$.

Proof: Set $K = \{3A, 3B, 4X\}$ where $X \in \{A, B, C, D, E, F\}$. If $nX \in K$ then $rank(Co_2 : nX) = 2$ by Lemmas 5 to 12.

Again direct computations using $\mathbb{GAP}$ and from the results of Ganief and Moori [8] we get that $Co_2 \cong \langle a, b\rangle$ where $a \in 2A$ and $b \in \{5A, 5B, 6A, 6B, 6E, 6F, 7A, 8A, 8B, 8C, 8D, 8F, 9A, 11A, 12A, 23A, 23B\}$. Now for group Co_2 we have the following power maps $(12A)^2 = 6B$, $(12B)^2 = 6A$, $(12D)^2 = 6E$, $(12E)^2 = 6B$, $(12F)^2 = 6E$, $(12G)^2 = 6A$, $(12H)^2 = 6E$, $(14A)^2 = 7A$, $(14B)^2 = 7A$, $(14C)^2 = 7A$, $(15A)^3 = 5B$, $(15B)^3 = 5A$, $(15C)^3 = 5A$, $(16A)^2 = 8D$, $(16B)^2 = 8C$, $(18A)^2 = 9A$, $(20A)^2 = 10A$, $(20B)^2 = 10C$, $(24A)^2 = 12C$, $(24B)^2 = 12B$, $(28A)^4 = 7A$, $(30A)^2 = 15A$, $(30B)^2 = 15B$ and $(30C)^2 = 15C$. Using the above power maps together with information from [8], we obtain that Co_2 is $(2A, nX, mY)$-generated for $nX \notin \{1A, 2A, 2B, 2C\}$ with appropriate mZ. Now applying Lemma 3, Co_2 is $(nX, nX, (mZ)^2)$-generated for $nX \notin \{1A, 2A, 2B, 2C\}$. Hence $\text{rank}(Co_2 : nX) = 2$ where $nX \notin \{1A, 2A, 2B, 2C\}$.

3 Ranks of Co_3

The smallest *Conway* group Co_3 is a sporadic simple group of order $2^{10}.3^7.5^3.7.11.23$ with 14 conjugacy classes of maximal subgroups. The group Co_3 has 42 conjugacy classes of its elements. It has two conjugacy classes of involutions, namely $2A$ and $2B$. For basic properties of Co_3 we refer readers to [2], [3] and [7].

Lemma 14 $\text{rank}(Co_3 : 2X) = 3$ *where* $X \in \{A, B\}$.

Proof: Wolder [22] and Ganief and Moori [8] proved that Co_3 is a Hurwitz group by showing that Co_3 is $(2B, 3C, 7A)$-generated. So again by Corollary 2, we have $rank(Co_3 : 2B) \leq 3$. But $\text{rank}(Co_3 : 2B) = 2$ is not possible, because if $\langle x, y \rangle = Co_3$ for some $x, y \in 2B$ then $Co_3 \cong D_{2n}$ with $o(xy) = n$. Thus $\text{rank}(Co_3 : 2B) = 3$.

Now for the rank of involution $2A$ in Co_3 we consider the triple $(2A, 3C, 23A)$. Using the character table of Co_3 we compute that $\Delta_{Co_3}(2A, 3C, 23A) = 46$. The only maximal subgroup of Co_3 containing elements of order 23, up to isomorphism, is M_{23}. But the conjugacy class $3C$ has empty intersection with M_{23}. Thus Co_3 contains no proper $(2A, 3C, 23A)$-subgroup and we get $\Delta^*_{Co_3}(2A, 3C, 23A) = \Delta_{Co_3}(2A, 3C, 23A) > 0$. Now the result follows again applying Corollary 2. □

Next we deal with the non-involution conjugacy classes of Co_3.

Lemma 15 *The group Co_3 is $(3A, 3A, 15A)$-generated.*

Proof: The only maximal subgroups of Co_3 having non-empty intersection with the classes $3A$ and $15A$, up to isomorphisms, are $M_1 \cong McL{:}2$, $M_2 \cong 2{\cdot}S_6(2)$, $M_3 \cong U_3(5){:}S_3$ and $M_4 \cong 3^{1+4}_{+}{:}4S_6$. By considering the permutation character values of Co_3 on these maximal subgroups and their fusion maps into Co_3 we obtain that $\Sigma_{M_1}(3A, 3A, 15A) = 6 = \Sigma_{M_3}(3A, 3A, 15A)$ and $\Sigma_{M_2}(3A, 3A, 15A) = 0 = \Sigma_{M_4}(3A, 3A, 15A)$. Hence

$$\begin{aligned}\Delta^*_{Co_3}(3A, 3A, 15A) &\geq \Delta_{Co_3}(3A, 3A, 15A) - \Sigma_{M_1}(3A, 3A, 15A) - \Sigma_{M_2}(3A, 3A, 15A) \\ &\quad -\Sigma_{M_3}(3A, 3A, 15A) - \Sigma_{M_4}(3A, 3A, 15A) \\ &= 46 - 1(6) - 1(6) > 0.\end{aligned}$$

This concludes that Co_3 is $(3A, 3A, 15A)$-generated. □

Corollary 16 $\text{rank}(Co_3 : 3A) = 2$.

Proof: From the previous lemma we know that Co_3 is $(3A, 3A, 15A)$-generated and so we get that $rank(Co_3 : 3A) = 2$.

Lemma 17 *The group Co_3 is $(4A, 4A, 23A)$-generated.*

Proof: The only maximal subgroup of Co_3 which has an order divisible by 23 is M_{23} and $M_{23} \cap 4A = \emptyset$. Hence $\Delta^*_{Co_3}(4A, 4A, 23A) = \Delta_{Co_3}(4A, 4A, 23A) = 414 > 0$. This shows that Co_3 is $(4A, 4A, 23A)$-generated. □

Lemma 18 *The group Co_3 is $(4B, 4B, 23A)$-generated.*

Proof: The only maximal subgroups of Co_3 which has an order divisible by 23 is M_{23}. The $4a$ and $23a$ are the only classes of M_{23} which fuse to $4B$ and $23A$ classes of Co_3 respectively. This implies that

$$\begin{aligned}\Delta^*_{Co_3}(4B, 4B, 23A) &\geq \Delta_{Co_3}(4B, 4B, 23A) - \Sigma_{M_{23}}(4A, 4B, 23A) \\ &= 174846 - 7866 > 0,\end{aligned}$$

proving that $(4B, 4B, 23A)$ is a generating triple for Co_3. □

Lemma 19 rank($Co_3 : 3B$) $= 2$.

Proof: The group Co_3 is $(3B, 3B, 23A)$-generated (see [8], Corollary 3.2). Hence rank($Co_3 : 3B$) $= 2$. □

Theorem 20 *If* $nX \notin \{1A, 2A, 2B\}$ *then* rank($Co_3 : nX$) $= 2$.

Proof: If $nX \in \{3A, 3B, 4A, 4B\}$ then $rank(Co_3 : nX) = 2$ by the above Lemmas 14 to 19.

Direct computation using GAP and results from Ganief and Moori [8] show that $Co_3 = \langle a, b\rangle$ where $a \in 2A$, and $b \in nX$ with $nX \in \{3C, 6A, 6B, 6C, 6D, 8A, 8B, 8C, 9A, 9B\}$. Since the power maps of Co_3 yields $(6E)^2 = 3C$, $(10A)^2 = 5A$, $(10B)^2 = 5B$, $(12A)^2 = 6A = (12B)^2$, $(12C)^2 = 6C$, $(14A)^2 = 7A$, $(15A)^3 = 5A$, $(15B)^3 = 5B$, $(18A)^3 = 6B$, $(20A)^4 = 5A = (20B)^4$, $(21A)^3 = 7A$, $(22A)^2 = 11B$, $(22B)^2 = 11A$, $(24A)^4 = 6A = (24B)^4$ and $(30A)^6 = 5A$, we have $Co_3 = \langle a, c\rangle$ where $c \in \{6E, 10A, 10B, 12A, 12B, 12C, 14A, 15A, 15B, 18A, 20A, 20B, 21A, 22A, 22B, 24A, 24B, 30A\}$.

Therefore Co_3 is $(2A, nX, mY)$-generated where $nX \in \{3C, 6A, 6B, 6C, 6D, 6E, 8A, 8B, 8C, 9A, 9B, 10A, 10B, 12A, 12B, 12C, 14A, 15A, 15B, 18A, 20A, 20B, 21A, 22A, 22B, 24A, 24B, 30A\}$ with appropriate mY. Hence rank($Co_3 : nX$) $= 2$ where $nX \notin \{1A, 2A, 2B\}$. □

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