On the ranks of the Harada-Norton sporadic simple group HN

Mohammad Ali Faya Ibrahim *
Department of Mathematics,
King Khalid University,
P.O. Box 9004, Abha,
Saudi Arabia.

Abstract

Let G be a finite group and X a conjugacy class of G. We denote rank(G:X) to be the minimum number of elements of X generating G. In the present article, we determine the ranks for the Harada-Norton sporadic group HN. Computations were carried with the aid of computer algebra system $\mathbb{GAP}$ [21].

1 Introduction and Preliminaries

There has recently been some interest in generation of simple groups by their conjugate involutions. It is well known that sporadic simple groups are generated by three conjugate involutions (see [7]). If a group $G = \langle a, b \rangle$ is perfect and $a^2 = b^3 = 1$ then clearly G is generated by three conjugate involutions a, a^b and a^{b^2} (see [8]). Moori [17] proved that the Fischer group Fi_{22} can be generated by three conjugate involutions. The work of Liebeck and Shalev [14] show that all but finitely many classical groups can be generated by three involutions. The generation of a simple group by its conjugate elements in this context is of some interest.

Suppose that G is a finite group and $X \subseteq G$. We denote the rank of X in G by rank(G:X), the minimum number of elements of X generating G. This paper focuses on the determination of rank(G:X) where X is a conjugacy class of G and G is a sporadic simple group.

Moori in [15], [16] and [17] proved that rank(Fi_{22}:$2A$) $\in \{5, 6\}$ and rank(Fi_{22}:$2B$) = rank(Fi_{22}:$2C$) = 3 where $2A, 2B$ and $2C$ are the conjugacy classes of involutions of the smallest Fischer group Fi_{22} as presented in the $\mathbb{ATLAS}$ [5]. The work of Hall

*E-Mail: mafibraheem@kku.edu.sa

and Soicher [11] show that rank(Fi_{22}:$2A$) = 6. Moori in [18] determined the ranks of the *Janko* groups J_1, J_2 and J_3. More recently, in a series of papers [1, 2, 3, 12] with F. Ali, the author investigated the ranks for the sporadic group HS, McL, Co_1 Co_2, Co_3, Ru, Suz and Th. In the present article we continue the study on the ranks of sporadic simple groups and determine the ranks for the Harada Norton sporadic simple groups HN.

For basic properties of HN, character tables of the groups HN and their maximal subgroups etc. we use $\mathbb{ATLAS}$ [5] and [21]. For detailed information about the computational techniques used in this paper the reader is encouraged to consult [1], [2], [17], and [18].

Next we discuss some background material and introduce the notation. We adopt the same notation as in the above mentioned papers. In particular, if G is a finite group, $C_1, C_2, \cdots, C_k$ are the conjugacy classes of its elements and g_k is a fixed representative of C_k, then $\Delta_G(C_1, C_2, \cdots, C_k)$ denotes the number of distinct tuples $(g_1, g_2, \cdots, g_{k-1}) \in (C_1 \times C_2 \times \cdots \times C_{k-1})$ such that $g_1 g_2 \cdots g_{k-1} = g_k$. It is well known that $\Delta_G(C_1, C_2, \cdots, C_k)$ is the structure constant of G for the conjugacy classes $C_1, C_2, \cdots, C_k$ and can be computed from the character table of G (see [13], p.45) by the following formula

$$\Delta_G(C_1, C_2, \cdots, C_k) = \frac{|C_1||C_2| \cdots |C_{k-1}|}{|G|} \times \sum_{i=1}^{m} \frac{\chi_i(g_1)\chi_i(g_2)\cdots\chi_i(g_{k-1})\overline{\chi_i(g_k)}}{[\chi_i(1_G)]^{k-2}}$$

where $\chi_1, \chi_2, \cdots, \chi_m$ are the irreducible complex characters of G. Also, $\Delta^*_G(C_1, C_2, \cdots, C_k)$ denotes the number of distinct tuples $(g_1, g_2, \cdots, g_{k-1}) \in (C_1 \times C_2 \times \cdots \times C_{k-1})$ such that $g_1 g_2 \cdots g_{k-1} = g_k$ and $G = \langle g_1, g_2, \cdots, g_{k-1}\rangle$. If $\Delta^*_G(C_1, C_2, \cdots, C_k)0$, then we say that G is $(C_1, C_2, \cdots, C_k)$-generated. If H any subgroup of G containing the fixed element $g_k \in C_k$, then $\Sigma_H(C_1, C_2, \cdots, C_{k-1}, C_k)$ denotes the number of distinct tuples $(g_1, g_2, \cdots, g_{k-1}) \in (C_1 \times C_2 \times \cdots \times C_{k-1})$ such that $g_1 g_2 \cdots g_{k-1} = g_k$ and $\langle g_1, g_2, \cdots, g_{k-1}\rangle \leq H$ where $\Sigma_H(C_1, C_2, \cdots, C_k)$ is obtained by summing the structure constants $\Delta_H(c_1, c_2, \cdots, c_k)$ of H over all H-conjugacy classes $c_1, c_2, \cdots, c_{k-1}$ satisfying $c_i \subseteq H \cap C_i$ for $1 \leq i \leq k-1$.

The $\mathbb{ATLAS}$ serves as a valuable source of information and we use the Atlas notation for conjugacy classes, maximal subgroups, character tables, permutation characters, etc. A general conjugacy class of elements of order n in G is denoted by nX. For examples, $2A$ represents the first conjugacy class of involutions in a group G. We will use the maximal subgroups and the permutations characters of HN on the conjugates (right cosets) of the maximal subgroups listed in the $\mathbb{ATLAS}$ [5] extensively.

The following results will be crucial in determining the ranks of finite groups.

Lemma 1 *(Moori [18]) Let G be a finite simple group such that G is (lX, mY, nZ)-generated. Then G is $(\underbrace{lX, lX, \cdots, lX}_{m-times}, (nZ)^m)$-generated.*

Corollary 2 *Let G be a finite simple group such that G is (lX, mY, nZ)-generated, then* $\mathrm{rank}(G : lX) \leq m$.

Proof: Immediately follows from Lemma 1. □

Lemma 3 *(Conder et al. [6]) Let G be a simple $(2X, mY, nZ)$-generated group. Then G is $(mY, mY, (nZ)^2)$-generated.*

2 Ranks for the group HN

The Harada-Norton group HN is a sporadic simple group of order

$$273030912000000 = 2^{14} \times 3^6 \times 5^6 \times 7 \times 11 \times 19$$

with 14 conjugacy classes of maximal subgroups. It has 54 conjugacy classes of its elements including two involutions, namely $2A$ and $2B$. For an element $g \in 5A$ in the Monster group M, we have $C_{\mathrm{M}}(g) \cong 5 \times \mathrm{HN}$. Norton [19] constructed HN as a permutation group on 1140000 points. For basic properties of HN and computational techniques, the reader is encouraged to consult [1], [4], [10], [19] and [22].

We now compute the rank of each conjugacy class of HN.

It is well known that every sporadic simple group can be generated by three involutions (see [8]). In the following lemmas we prove that HN can be generated by three conjugate involutions $a, b, c \in 2X$ where $X \in \{A, B\}$.

Lemma 4 *The sporadic group HN is $(2X, 2X, 2X, 40A)$-generated where $X \in \{A, B\}$.*

Proof: First we treat the case $X = A$. Simple computations show that the structure constant $\Delta_{HN}(2A, 2A, 2A, 40A) = 2240$. By looking into the maximal subgroups of HN given in the $\mathbb{ATLAS}$ [5], we observe that the only maximal subgroups having non-empty intersection with the conjugacy classes $2A$ and $40A$ are, up to isomorphism, $2{\cdot}HS.2, (D_{10}{\times}U_3(5)){\cdot}2, (A_6{\times}A_6){\cdot}D_8$ and $5_+^{1+4}{:}2_-^{1+4}.5.4$. A fixed element of order 40 in HN is contained in precisely two conjugate of $(A_6 \times A_6) \cdot D_8$ and in a unique conjugate of each $2{\cdot}HS.2$, $(D_{10}{\times}U_3(5)){\cdot}2$ and $5_+^{1+4} : 2_-^{1+4}.5.4$. Our computations show that $\Sigma_{2\cdot HS.2}(2A, 2A, 2A, 40A) = 960$, and $\Sigma_{(D_{10}\times U_3(5))\cdot 2}(2A, 2A, 2A, 40A) = \Sigma_{(A_6\times A_6)\cdot D_8}(2A, 2A, 2A, 40A) = \Sigma_{5_+^{1+4}:2_-^{1+4}.5.4}(2A, 2A, 2A, 40A) = 0$. Therefore,

$$\begin{aligned} \Delta^*_{\mathrm{HN}}(2A, 2A, 2A, 40A) &\geq \Delta_{\mathrm{HN}}(2A, 2A, 2A, 40A) - \Sigma_{2\cdot HS.2}(2A, 2A, 2A, 40A) \\ &= 2240 - 960 > 0. \end{aligned}$$

This concludes that HN is $(2A, 2A, 2A, 40A)$-generated.

Next consider the case $X = B$. For this case $2{\cdot}HS.2$, $(D_{10}{\times}U_3(5)){\cdot}2$, $(A_6{\times}A_6){\cdot}D_8$ and $5_+^{1+4}{:}2_-^{1+4}.5.4$ are, up to isomorphisms, the only maximal subgroups of the group HN that may contain $(2B, 2B, 2B, 40A)$-generated proper subgroups of HN. We calculate $\Delta_{HN}(2B, 2B, 2B, 40A) = 1461574400$, $\Sigma_{2\cdot HS.2}(2B, 2B, 2B, 40A) = 1107200$, $\Sigma_{(D_{10}\times U_3(5))\cdot 2}(2B, 2B, 2B, 40A) = 0$, $\Sigma_{(A_6\times A_6)\cdot D_8}(2B, 2B, 2B, 40A) = 0$ and $\Sigma_{5_+^{1+4}:2_-^{1+4}.5.4}$ $(2B, 2B, 2B, 40A) = 0$. Thus

$$\begin{aligned}\Delta^*_{\mathrm{HN}}(2B, 2B, 2B, 40A) &\geq \Delta_{\mathrm{HN}}(2B, 2B, 2B, 40A) - \Sigma_{2\cdot HS.2}(2B, 2B, 2B, 40A)\\ &= 1461574400 - 1107200 > 0.\end{aligned}$$

Hence, HN is $(2X, 2X, 2X, 40A)$-generated where $X \in \{A, B\}$.

Corollary 5 $\mathrm{rank}(HN{:}\,2X) = 3$ *where* $X \in \{A, B\}$.

Proof: Let $X \in \{A, B\}$. Then the previous lemma implies that $rank(\mathrm{HN} : 2X) \leq 3$. But the case $rank(HN : 2X) = 2$ is not possible since if there are $x, y \in 2X$ such that $HN =< x, y >$, then $\mathrm{HN} \cong D_{2n}$ where $n = o(xy)$. This shows that $\mathrm{rank}(\mathrm{HN} : 2A) = 3 = \mathrm{rank}(\mathrm{HN} : 2B)$. □

Lemma 6 *Let nX be a conjugacy class of he sporadic group HN such that $nX \notin \{1A, 2A, 2B\}$. Then HN is $(2A, nX, 40A)$-generated.*

Proof: The investigation of the $(2A, nX, 40A)$-generation of the group HN will require knowledge of all the maximal subgroups of HN having elements of order 40. They are, up to isomorphisms, $H_1 \cong 2{\cdot}HS.2$, $H_2 \cong (D_{10}{\times}U_3(5)){\cdot}2$, $H_3 \cong (A_6{\times}A_6){\cdot}D_8$ and $H_4 \cong 5_+^{1+4} : 2_-^{1+4}.5.4$. Also a fixed element $z \in 40A$ is contained in precisely two conjugate of H_3 and in a unique conjugate of each of H_1, H_2 and H_4. Now, our main proof will consider a number of cases.

Case 1: Let $nX \in T_1 = \{9A, 15B, 15C, 19A, 19B, 21A, 30B, 30C\}$. From the fusion maps of the maximal subgroups of HN into the group HN, we observe that there is no maximal subgroup of HN that may contain $(2A, nX, 40A)$-generated proper subgroups and hence no proper subgroup of HN is $(2A, nX, 40A)$-generated. Using Table I, we obtain that

$$\Delta^*_{\mathrm{HN}}(2A, nX, 40A) = \Delta_{\mathrm{HN}}(2A, nX, 40A) > 0.$$

Therefore, the group HN is $(2A, nX, 40A)$-generated where $nX \in T_1$.

Case 2: $nX \in T_2$=$\{11A, 22A\}$. The only maximal subgroup of HN having non-empty intersection with the conjugacy classes in the triple $(2A, nX, 40A)$ where $nX \in T_2$ is isomorphic to H_1. It is evident from Table I that

$$\begin{aligned}\Delta^*_{\mathrm{HN}}(2A, 11A, 40A) &= \Delta_{\mathrm{HN}}(2A, 11A, 40A) - \Sigma_{H_1}(2A, 11A, 40A) > 0\\ \Delta^*_{\mathrm{HN}}(2A, 22A, 40A) &= \Delta_{\mathrm{HN}}(2A, 22A, 40A) - \Sigma_{H_1}(2A, 22A, 40A) > 0.\end{aligned}$$

Hence HN is $(2A, 11A, 40A)-$, and $(2A, 22A, 40A)$-generated.

Case 3: $nX \in T_3 = \{35A, 35B\}$. Amongst the maximal subgroups of HN, the only maximal subgroup with non-empty intersection with the classes in the triple $(2A, nX, 40A)$ for $nX \in T_3$, is isomorphic to H_2. We can easily see that for $nX \in T_3$, we have $\Delta^*_{\mathrm{HN}}(2A, nX, 40A) = \Delta_{\mathrm{HN}}(2A, nX, 40A) > 0$, proving that $(2A, 35A, 40A)$ and $(2A, 35B, 40A)$ are the generating triples for HN.

Case 4: $nX \in T_4 = \{10D, 10E, 20D, 20E, 25A, 25B\}$. H_3 is the only maximal subgroup of HN with non-empty intersection with any conjugacy class in the triple $(2A, nX, 40A)$ where $nX \in T_4$. We calculate that in each case $\Delta^*_{\mathrm{HN}}(2A, nX, 40A) = \Delta_{\mathrm{HN}}(2A, nX, 40A) - 2\Sigma_{H_3}(2A, nX, 40A) > 0$. Hence HN is $(2A, nX, 40A)$-generated where $nX \in T_4$.

Case 5: $nX \in T_5 = \{3B, 6C, 12C\}$. H_4 is the only maximal subgroup of HN which meets the classes in the triple $(2A, nX, 40A)$ in this case. But $\Sigma_{H_4}(2A, nX, 40A) = 0$ for each $nX \in T_5$. Thus $\Delta^*_{\mathrm{HN}}(2A, nX, 40A) = \Delta_{\mathrm{HN}}(2A, nX, 40A) > 0$, proving that $(2A, nX, 40A)$ is the generating triples for HN.

Case 6: $nX \in T_6 = \{7A, 12B, 14A, 30A\}$. The maximal subgroups of HN with non-empty intersection with all the classes in the triple $(2A, nX, 40A)$ for $nX \in T_6$ are, up to isomorphisms, H_1 and H_2. We calculate, using Table I, $\Delta^*_{\mathrm{HN}}(2A, nX, 40A) \geq \Delta_{\mathrm{HN}}(2A, nX, 40A) - \Sigma_{H_1}(2A, nX, 40A) - \Sigma_{H_2}(2A, nX, 40A) > 0$ and generation by the triple $(2A, nX, 40A)$ for $nX \in T_6$ follows.

Case 7: $nX \in T_7 = \{10G, 10H\}$. Amongst the maximal subgroups of HN, H_1 and H_3 are the only maximal subgroup which may contain $(2A, 10G, 40A)$- and $(2A, 10H, 40A)$-generated proper subgroups. But

$$\Sigma_{H_1}(2A, 10G, 10H) = 0 = \Sigma_{H_3}(2A, 10H, 40A)$$

$$\Sigma_{H_1}(2A, 10G, 10H) = 0 = \Sigma_{H_3}(2A, 10H, 40A)$$

Therefore,

$$\begin{aligned} \Delta^*_{\mathrm{HN}}(2A, 10G, 40A) &= \Delta_{\mathrm{HN}}(2A, 10G, 40A) = 25257687040 \\ \Delta^*_{\mathrm{HN}}(2A, 10H, 40A) &= \Delta_{\mathrm{HN}}(2A, 10H, 40A) = 25257687040 \end{aligned}$$

proving the generation of HN by these triples.

Case 8: Consider the triple $(2A, 4C, 40A)$. The only maximal subgroups of HN, having non-empty intersection with the classes in this triple are isomorphic to H_3 and H_4. We calculate using Table I that, $\Sigma_{H_3}(2A, 4C, 10H) = 0 = \Sigma_{H_4}(2A, 4C, 40A)$ and we obtain

$$\Delta^*_{\mathrm{HN}}(2A, 4C, 40A) = \Delta_{\mathrm{HN}}(2A, 4C, 40A) = 15744000.$$

Hence HN is $(2A, 4C, 40A)$-generated.

Case 9: Consider the triple $(2A, 5C, 40A)$ and $(2A, 5D, 40A)$. The only maximal subgroups of HN with non-empty intersection with the classes in these triples are isomorphic to H_2 and H_3. We calculate $\Sigma_{H_2}(2A, 5C, 40A) = 576 = \Sigma_{H_2}(2A, 5D, 40A)$ and $\Sigma_{H_3}(2A, 5C, 40A) = 0 = \Sigma_{H_3}(2A, 5D, 40A)$. Hence $(2A, 5C, 40A)$ and $(2A, 5D, 40A)$ are the generating triples for HN since $\Delta_{\mathrm{HN}}(2A, 5C, 40A) = 690016 = \Delta_{HN}(2A, 5D, 40A)$.

Case 10: $nX \in T_8 = \{5B, 10A, 10C, 20A, 20B\}$. The maximal subgroups of HN with non-empty intersection with all the classes in the triple $(2A, nX, 40A)$ for $nX \in T_8$ are, up to isomorphisms, H_1, H_2 and H_3. Using Table I, we compute

$$\begin{aligned}\Delta^*_{\mathrm{HN}}(2A, nX, 40A) \geq{}& \Delta_{\mathrm{HN}}(2A, nX, 40A) - \Sigma_{H_1}(2A, nX, 40A)\\ &-\Sigma_{H_2}(2A, nX, 40A) - 2\Sigma_{H_3}(2A, nX, 40A) > 0\end{aligned}$$

proving generation of HN by the triple $(2A, nX, 40A)$ for $nX \in T_8$ follows.

Case 11: $nX \in T_9 = \{3A, 6A, 6B, 12A, 15B\}$. H_1, H_2 and H_4 are only maximal subgroups of HN that may contain $(2A, nX, 40A)$-generated proper subgroups where $nX \in T_9$. Thus,

$$\begin{aligned}\Delta^*_{\mathrm{HN}}(2A, nX, 40A) \geq{}& \Delta_{\mathrm{HN}}(2A, nX, 40A) - \Sigma_{H_1}(2A, nX, 40A)\\ &-\Sigma_{H_2}(2A, nX, 40A) - \Sigma_{H_4}(2A, nX, 40A) > 0\end{aligned}$$

and generation of HN by the triple $(2A, nX, 40A)$ where $nX \in T_9$ follows.

Case 12: Consider the triple $(2A, 8B, 40A)$. For classes in this triple the maximal subgroups of HN which have non-empty intersection are, up to isomorphisms, H_1, H_3 and H_4. Therefore by Table I, we calculate,

$$\begin{aligned}\Delta^*_{\mathrm{HN}}(2A, 8B, 40A) \geq{}& \Delta_{\mathrm{HN}}(2A, 8B, 40A) - \Sigma_{H_1}(2A, 8B, 40A)\\ &-2\Sigma_{H_3}(2A, 8B, 40A) - \Sigma_{H_4}(2A, 8B, 40A) > 0.\end{aligned}$$

Thus HN is $(2A, 8B, 40A)$-generated.

Case 13: $nX \in T_{10} = \{4A, 4B, 5A, 5E, 8A, 10A, 10F, 20C, 40A, 40B\}$. The only maximal subgroups of HN with non-empty intersection with all the classes in the triple $(2A, nX, 40A)$ where $nX \in T_{10}$ are, up to isomorphisms, H_1, H_2, H_3 and H_4. Our calculations give, (see Table I) in each case we obtain

$$\begin{aligned}\Delta^*_{\mathrm{HN}}(2A, nX, 40A) \geq{}& \Delta_{\mathrm{HN}}(2A, nX, 40A) - \Sigma_{H_1}(2A, nX, 40A) - \Sigma_{H_2}(2A, nX, 40A)\\ &-2\Sigma_{H_3}(2A, nX, 40A) - \Sigma_{H_4}(2A, nX, 40A) > 0.\end{aligned}$$

Therefore HN $(2A, nX, 40A)$-generated for each $nX \in T_{10}$. This completes the proof of lemma. □

We now give the main result of the paper.

Theorem 7 *Let nX be a conjugacy class of the group HN. such that . Then*

(i) $rank(HN\colon nX) = 3$ *if* $nX \in \{2A, 2B\}$

(ii) $rank(HN\colon nX) = 2$ *if* $nX \notin \{1A, 2A, 2B\}$

Proof: We proved in the previous lemma that the group HN is $(2A, nX, 40A)$-generated for $nX \notin \{1A, 2A, 2B\}$. Applying Lemma 3, we obtain that the group HN is $(nX, nX, (40A)^2)$-generated and hence $rank(HN : nX) = 2$ where $nX \notin \{1A, 2A, 2B\}$. The proof now follows by Corollary 5. □

Table I: Structure Constants

tX	3A	3B	4A	4B	4C	5A
$\Delta_{\mathrm{HN}}(2A, nX, 40A)$	240	299840	741440	8013680	15744000	128
$\Sigma_{H_1}(2A, nX, 40A)$	0	-	1600	28800	-	0
$\Sigma_{H_2}(2A, nX, 40A)$	0	-	0	9440	-	48
$\Sigma_{H_3}(2A, nX, 40A)$	-	-	0	0	0	0
$\Sigma_{H_4}(2A, nX, 40A)$	0	0	0	1360	0	0

tX	5B	5C	5D	5E	6A	6B
$\Delta_{\mathrm{HN}}(2A, nX, 40A)$	320	690016	690016	37597104	113997360	809681840
$\Sigma_{H_1}(2A, nX, 40A)$	0	-	-	0	114080	157920
$\Sigma_{H_2}(2A, nX, 40A)$	0	576	576	1344	240	4160
$\Sigma_{H_3}(2A, nX, 40A)$	0	0	0	0	-	-
$\Sigma_{H_4}(2A, nX, 40A)$	-	-	-	0	0	0

tX	6C	7A	8A	8B	9A	10A
$\Delta_{\mathrm{HN}}(2A, nX, 40A)$	1980103680	1546088400	42014895040	65653408000	368894940480	52419360
$\Sigma_{H_1}(2A, nX, 40A)$	-	0	547840	684800	-	0
$\Sigma_{H_2}(2A, nX, 40A)$	-	0	112320	-	-	0
$\Sigma_{H_3}(2A, nX, 40A)$	-	-	0	0	-	0
$\Sigma_{H_4}(2A, nX, 40A)$	0	-	0	3200	-	-

tX	10B	10C	10D	10E	10F	10G
$\Delta_{\mathrm{HN}}(2A, nX, 40A)$	185938536	425098560	673315840	673315840	26863501600	25257687040
$\Sigma_{H_1}(2A, nX, 40A)$	10104	0	-	-	0	0
$\Sigma_{H_2}(2A, nX, 40A)$	5016	6240	-	-	3840	-
$\Sigma_{H_3}(2A, nX, 40A)$	0	0	0	0	0	0
$\Sigma_{H_4}(2A, nX, 40A)$	0	-	-	-	0	-

tX	10H	11A	12A	12B	12C	14A
$\Delta_{\mathrm{HN}}(2A, nX, 40A)$	25257687040	555614906080	51880630720	116731738880	1895848980480	348241723840
$\Sigma_{H_1}(2A, nX, 40A)$	0	0	414720	0	-	1811760
$\Sigma_{H_2}(2A, nX, 40A)$	-	-	33280	33280	-	51840
$\Sigma_{H_3}(2A, nX, 40A)$	0	-	-	-	-	-
$\Sigma_{H_4}(2A, nX, 40A)$	-	-	0	-	0	-

tX	15A	15B	15C	19A	19B	20A
$\Delta_{\mathrm{HN}}(2A, nX, 40A)$	7923376064	296506368000	296506368000	756318114720	756318114720	42653088000
$\Sigma_{H_1}(2A, nX, 40A)$	0	-	-	-	-	885760
$\Sigma_{H_2}(2A, nX, 40A)$	144	-	-	-	-	12480
$\Sigma_{H_3}(2A, nX, 40A)$	-	-	-	-	-	0
$\Sigma_{H_4}(2A, nX, 40A)$	0	-	-	-	-	-

tX	20B	20C	20D	20E	21A	22A
$\Delta_{\mathrm{HN}}(2A, nX, 40A)$	42653088000	168069677296	667178496000	667178496000	619086986240	555614906080
$\Sigma_{H_1}(2A, nX, 40A)$	885760	0	-	-	-	0
$\Sigma_{H_2}(2A, nX, 40A)$	12480	53616	-	-	-	-
$\Sigma_{H_3}(2A, nX, 40A)$	0	0	0	0	-	-
$\Sigma_{H_4}(2A, nX, 40A)$	-	10640	-	-	-	-

tX	25A	25B	30A	30B	30C	35A
$\Delta_{\mathrm{HN}}(2A, nX, 40A)$	427018524000	427018524000	71291952576	296506368000	296506368000	222926845856
$\Sigma_{H_1}(2A, nX, 40A)$	-	-	397728	-	-	-
$\Sigma_{H_2}(2A, nX, 40A)$	-	-	2496	-	-	7776
$\Sigma_{H_3}(2A, nX, 40A)$	0	0	-	-	-	-
$\Sigma_{H_4}(2A, nX, 40A)$	-	-	-	-	-	-

tX	35B	40A	40B
$\Delta_{\mathrm{HN}}(2A, nX, 40A)$	222926845856	168069677296	168069677296
$\Sigma_{H_1}(2A, nX, 40A)$	-	0	0
$\Sigma_{H_2}(2A, nX, 40A)$	7776	5616	5616
$\Sigma_{H_3}(2A, nX, 40A)$	-	0	0
$\Sigma_{H_4}(2A, nX, 40A)$	-	0	0

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