On the ranks of certain sporadic simple groups by Suzuki, Thompson and Rudvalis

Mohammed Ali Faya Ibrahim *
Department of Mathematics, Faculty of Science,
King Khalid University, P.O. Box 9004,
Abha, Saudi Arabia

Abstract

Let G be a finite group and X a conjugacy class of G. We denote $rank(G : X)$ to be the minimum number of elements of X generating G. In the present paper we determine the ranks for the three sporadic simple groups Suzuki group Suz, Thompson group Th and Rudvalis group Ru. Computations were carried with the aid of computer algebra system $\mathbb{GAP}$ [18].

*E-Mail: mafibraheem@kku.edu.sa

1 Introduction and Preliminaries

Let G be a finite group and $X \subseteq G$. We denote rank of X in G by $rank(G:X)$, the minimum number of elements of X generating G. This paper focuses on the determination of $rank(G:X)$ where X is a conjugacy class of G and G is a sporadic simple group.

Moori in [14], [15] and [16] showed that $rank(Fi_{22} : 2A) \in \{5, 6\}$ and $rank(Fi_{22} : 2B) = rank(Fi_{22} : 2C) = 3$ where $2A$, $2B$ and $2C$ are the conjugacy classes of involutions of the smallest Fischer group Fi_{22} as represented in the $\mathbb{ATLAS}$ [6]. Hall and Soicher in [10] proved that $rank(Fi_{22} : 2A) = 6$. In [17], Moori determined the ranks of the Janko groups J_1, J_2 and J_3.

This paper is intended as a sequel to the author's earlier papers on the ranks of the sporadic simple groups. In a series of papers the author determined the ranks of the sporadic simple groups HS, McL, Co_1, Co_2 and Co_3 (*cf.* [1, 2, 11]). In the present paper we compute the ranks of three sporadic simple groups, namely Suzuki group Suz, Thompson group Th and Rudvalis group Ru.

For basic properties of these groups, their character tables and their maximal subgroups we use $\mathbb{ATLAS}$ [6] and $\mathbb{GAP}$ [18]. For detailed information about the computational techniques used in this paper the reader is encouraged to consult [1, 2, 3, 4, 8, 16, 17].

In this paper we use the same notation as in [1], [2] and [14]. In particular, for a finite group G with $C_1, C_2, \ldots, C_k$ conjugacy classes of its elements and g_k a fixed representative of C_k, we denote $\Delta_G(C_1, C_2, \ldots, C_k)$ the number of distinct tuples $(g_1, g_2, \ldots, g_{k-1})$ with $g_i \in C_i$ such that $g_1 g_2 \ldots g_{k-1} = g_k$. It is well known that $\Delta_G(C_1, C_2, \ldots, C_k)$ is structure constant for the conjugacy classes $C_1, C_2, \ldots, C_k$ and can be easily computed from the character table of G (see [12], p.45) by the following formula

$$\Delta_G(C_1, C_2, \ldots, C_k) = \frac{|C_1||C_2|\ldots|C_{k-1}|}{|G|} \times \sum_{i=1}^{m} \frac{\chi_i(g_1)\chi_i(g_2)\ldots\chi_i(g_{k-1})\overline{\chi_i(g_k)}}{[\chi_i(1_G)]^{k-2}}$$

where $\chi_1, \chi_2, \ldots, \chi_m$ are the irreducible complex characters of G. Further let $\Delta^*_G(C_1, C_2, \ldots, C_k)$ denote the number of distinct tuples $(g_1, g_2, \ldots, g_{k-1})$ with $g_i \in C_i$ and $g_1 g_2 \ldots g_{k-1} = g_k$ such that $G =< g_1, g_2, \ldots, g_{k-1} >$. If $\Delta^*_G(C_1, C_2, \ldots, C_k) > 0$, then we say that G is $(C_1, C_2, \ldots, C_k)$-generated. If H is ¿a subgroup of G containing g_k and B is a conjugacy class of H

such that $g_k \in B$, then $\Sigma_H(C_1, C_2, \ldots, C_{k-1}, B)$ denotes the number of distinct tuples $(g_1, g_2, \ldots, g_{k-1})$ such that $g_i \in C_i$ and $g_1 g_2 \ldots g_{k-1} = g_k$ and $\langle g_1, g_2, \ldots, g_{k-1} \rangle \leq H$.

For the description of the conjugacy classes, the character tables, permutation characters and information on the maximal subgroups reader is referred to $\mathbb{ATLAS}$ [6]. A general conjugacy class of elements of order n in G is denoted by nX. For example $2A$ represents the first conjugacy class of involutions in a group G and $5AB$ represents the first two conjugacy classes of elements of order 5 in G respectively. We will use the maximal subgroups and the permutation characters on the conjugates (right cosets) of the maximal subgroups listed in the $\mathbb{ATLAS}$ extensively.

The following results will be crucial in determining the ranks of a finite group G.

Lemma 1 *(Moori [17]) Let G be a finite simple group such that G is (lX, mY, nZ)-generated. Then G is $(\underbrace{lX, lX, \ldots, lX}_{m-times}, (nZ)^m)$-generated.*

Corollary 2 *Let G be a finite simple group such that G is (lX, mY, nZ)-generated, then $rank(G : lX) \leq m$.*

Proof. The proof follows immediately from Lemma 1.

Lemma 3 *(Conder et al. [7]) Let G be a simple $(2X, mY, nZ)$-generated group. Then G is $(mY, mY, (nZ)^2)$-generated.*

We will employ results that, in certain situations, will effectively establish non-generation. They include Scott's theorem (*cf.* [7] and [19]) and Lemma 3.3 in [20] which we state here.

Lemma 4 *([20]) Let G be a finite centerless group and suppose lX, mY, nZ are G-conjugacy classes for which $\Delta^*(G) = \Delta^*_G(lX, mY, nZ) < |C_G(nZ)|$. Then $\Delta^*(G) = 0$ and therefore G is not (lX, mY, nZ)-generated.*
$\square$

2 Ranks of Suz

The Suzuki group Suz is a sporadic simple group of order

$$448345497600 = 2^{13}.3^7.5^2.7.11.13.$$

It is well known that Suz has exactly 43 conjugacy classes of its elements and 17 conjugacy classes of its maximal subgroups as listed in the $\mathbb{ATLAS}$ [6]. It has precisely two classes of involutions, namely $2A$ and $2B$ as represented in the $\mathbb{ATLAS}$. In the following lemma we prove that the Suzuki group Suz can be generated by three conjugate involutions.

Lemma 5 *Let* $X \in \{A, B\}$ *then* $rank(Suz : 2X) = 3$.

Proof. Direct computation in $\mathbb{GAP}$ using the character table of Suz shows that the structure constant $\Delta_{\mathrm{Suz}}(2A, 2A, 2A, 14A) = 5096$. The only maximal subgroups of Suz with elements of order 14 and having non-trivial intersection with classes $2A$ and $14A$ are, up to isomorphism, J_2:2 and $(A_4 \times PSL(3, 4))$:2. An easy computation reveals that

$$\Sigma_{J_2:2}(2A, 2A, 2A, 14A) = \Sigma_{(A_4 \times PSL(3,4)):2}(2A, 2A, 2A, 14A) = 0.$$

It follows that $\Delta^*_{\mathrm{Suz}}(2A, 2A, 2A, 14A) = \Delta_{\mathrm{Suz}}(2A, 2A, 2A, 14A) = 5096$. Thus Suz has no proper $(2A, 2A, 2A, 14A)$-generated subgroup and so is itself $(2A, 2A, 2A, 14A)$-generated. Now for the conjugacy class $2B$, we can similarly show that Suz is $(2B, 2B, 2B, 14A)$-generated.

Hence $rank(\mathrm{Suz} : 2X) \leq 3$ where $X \in \{A, B\}$ but $rank(\mathrm{Suz} : 2X) = 2$ is not possible, because if $\langle x, y\rangle = Suz$ for some $x, y \in 2X$ then Suz $\cong D_{2n}$ with $o(xy) = n$. Therefore $rank(\mathrm{Suz} : 2X) = 3$ where $X \in \{A, B\}$. □

Lemma 6 $rank(Suz : 3A) = 3$.

Proof. First we show that $rank(\mathrm{Suz} : 3A) > 2$ by proving that Suz is not $(3A, 3A, tX)$-generated for any conjugacy class tX of Suz. If Suz is $(3A, 3A, tX)$-generated then $\frac{1}{3} + \frac{1}{3} + \frac{1}{t} < 1$ and it follows that $t \geq 4$. Set $K = \{4A, 5A, 6A\}$. Using $\mathbb{GAP}$ [18] we see that $\Delta_{Suz}(3A, 3A, tX) = 0$ if $tX \notin K$ and for $tX \in K$ we have $\Delta_{\mathrm{Suz}}(3A, 3A, tX) < |C_{\mathrm{Suz}}(tX)|$. We get that

$$\Delta^*_{\mathrm{Suz}}(3A, 3A, tX) < \Delta_{\mathrm{Suz}}(3A, 3A, tX) < |C_{\mathrm{Suz}}(tX)|.$$

Applying Lemma 4, we obtain that $\Delta^*_{\mathrm{Suz}}(3A, 3A, tX) = 0$ for all tX with $t \geq 4$ and therefore Suz is not $(3A, 3A, tX)$-generated which means that $rank(\mathrm{Suz} : 3A) > 2$. Now since Suz is $(3A, 3C, 21A)$-generated (see [3]) the result follows by applying Corollary 2. $\square$

Lemma 7 *Let nX be a conjugacy class of Suz such that $nX \notin \{1A, 2A, 2B, 3A\}$. Then $rank(Suz : nX) = 2$.*

Proof. Our main proof considers the following three cases.

Case 1. Let $nX = 3B$. Here $M_{12}.2$ is the only maximal subgroup of Suz having non-empty intersection with the conjugacy classes $2B$, $3B$ and $11A$ of Suz. Our calculations give $\Delta_{\mathrm{Suz}}(2B, 3B, 11A) = 77$, and $\Sigma_{M12.2}(2B, 3B, 11A) = 11$. Since a fixed element of order 11 is contained in a unique conjugate of $M_{12}.2$ subgroup of Suz. Hence $\Delta^*_{\mathrm{Suz}}(2B, 3B, 11A) \geq 77 - 11 = 66 > 0$. Thus Suz is $(2B, 3B, 11A)$-generated and again by applying Lemma 3 we have $rank(\mathrm{Suz} : 3B) = 2$. This completes the proof.

Case 2. Let $nX \in K$ where $K = \{4A, 6A\}$. We observe that no maximal subgroup of Suz have non-empty intersection with the classes $2B$, $11A$ and for each $nX \in K$. Since $\Delta_{\mathrm{Suz}}(2B, 4A, 11A) = 22$ and $\Delta_{\mathrm{Suz}}(2B, 6A, 11A) = 638$, we obtain that $\Delta^*_{\mathrm{Suz}}(2B, nX, 11A) = \Delta_{\mathrm{Suz}}(2B, nX, 11A) > 0$ where $nX \in K$. Hence Suz is $(2B, nX, 11A)$-generated and by Lemma 3 we have $rank(\mathrm{Suz} : nX) = 2$ for each $nX \in K$.

Case 3. Let $nX \in \Omega$ where $\Omega = \{3C, 4BCD, 5AB, 6BCDE, 7A, 8A, 9AB, 10AB, 11A, 13AB, 12ABCDE, 14A, 15ABC, 18AB, 20A, 21AB, 24A\}$. Direct computations using $\mathbb{GAP}$ and from the results of [3] together with the power maps of the elements of Suz, we obtain that $\mathrm{Suz} = \langle a, b \rangle$ where $a \in 2A$ and $b \in \Omega$. Thus Suz is $(2A, nX, mY)$-generated for each $nX \in \Omega$ and with appropriate mY. Applying Lemma 3, we obtain that Suz is $(nX, nX, (mY)^2)$-generated with appropriate mY and so $rank(\mathrm{Suz} : nX) = 2$ for each $nX \in \Omega$. $\square$

We now summarize our results of this section in the following theorem:

Theorem 8 *Let Suz be the Suzuki's sporadic simple group. Then*

(i) $rank(Suz : nX) = 3$ if $nX \in \{2A, 2B, 3A\}$.

(ii) $rank(Suz : nX) = 2$ if $nX \notin \{1A, 2A, 2B, 3A\}$.

Proof. The proof follows from Lemmas 5, 6 and 7. $\square$

3 Ranks of Th

The Thompson group Th is a sporadic simple group of order

$$90745943887872000 = 2^{15}.3^{10}.5^3.7^2.13.19.31$$

It is well known fact that Th has exactly 16 conjugacy classes of maximal subgroups as listed in [13] and [6]. It has precisely one class of involution, namely $2A$ as represented in the $\mathbb{ATLAS}$ [6].

Lemma 9 $rank(Th : 2A) = 3$.

Proof. Linton in [13] and Ashrafi in [3] proved that Thompson group Th is $(2A, 3C, 7A)$-generated, that is Th is Hurwitz group. Thus Corollary 2 implies that $rank(\mathrm{Th} : 2A) \leq 3$. Since the Thompson group Th is not isomorphic to any Dihedral group D_{2n}, we must have $rank(\mathrm{Th} : 2A) = 3$. □

Lemma 10 *Let* nX *be a conjugacy class of Th such that* $nX \notin \{1A, 2A\}$. *Then* $rank(Th : nX) = 2$.

Proof. Direct computation using $\mathbb{GAP}$ and from the results of Ashrafi [3] together with information about the power maps, we can show that Th is $(2A, nX, mZ)$-generated for all $nX \notin \{1A, 2A\}$ with appropriate mZ. Now by Lemma 3, Th is $(nX, nX, (mZ)^2)$-generated for all $nX \notin \{1A, 2A\}$. Hence $rank(\mathrm{Th} : nX) = 2$ where $nX \notin \{1A, 2A\}$. □

We are now ready to state the main result of this section.

Theorem 11 *Let Th be the* T*hompson's sporadic simple group. Then*

(i) $rank(Th : 2A) = 3$.

(ii) $rank(Th : nX) = 2$ *where* $nX \notin \{1A, 2A\}$.

Proof. The proof follows from Lemmas 9 and 10.

4 Ranks of Ru

The Rudvalis group Ru is a sporadic simple group of order

$$145926144000 = 2^{14}.3^3.5^3.7.13.29$$

It is well known fact that Ru has exactly 15 conjugacy classes of maximal subgroups as listed in the $\mathbb{ATLAS}$ [6]. It has precisely two classes of involution, namely $2A$ and $2B$ as represented in the $\mathbb{ATLAS}$.

Theorem 12 *Let Ru be the Rudvalis's sporadic simple group. Then*

(i) $rank(Ru : nX) = 3$ *if* $nX \in \{2A, 2B\}$.

(ii) $rank(Ru : nX) = 2$ *if* $nX \notin \{1A, 2A, 2B\}$.

Proof. First suppose that $X \in \{A, B\}$. We know that Ru is $(2X, 3A, 29A)$-generated (see [8]). By applying Corollary 2 we obtain that $rank(\mathrm{Ru} : 2X) \leq 3$. But $rank(\mathrm{Ru} : 2X) = 2$ is not possible since Ru is not isomorphic to any Dihedral group D_{2n} and therefore $rank(\mathrm{Ru} : 2X) = 3$ where $X \in \{A, B\}$.

Next suppose that $nX \notin \{1A, 2A, 2B\}$ and we prove that Ru is $(2A, nX, 29A)$-generated. From the list of maximal subgroups of Ru given in the $\mathbb{ATLAS}$ we observe that no maximal subgroup of Ru intersects with the conjugacy classes $2A$, $29A$ and nX of Ru. Hence

$$\Delta^*_{\mathrm{Ru}}(2A, nX, 29A) \geq \Delta_{\mathrm{Ru}}(2A, nX, 29A) > 0 \text{ (see Table I).}$$

Hence Ru is $(2A, nX, 29A)$-generated and we obtain that $rank(\mathrm{Ru} : nX) = 2$ by applying Lemma 3. This completes the proof. $\square$

Table I: Structure Constants of Ru

nX	$3A$	$4A$	$4B$	$4C$	$4D$	$5A$
$\Delta_{\mathrm{Ru}}(2A, nX, 29A)$	203	29	174	551	1015	551
nX	$5B$	$6A$	$7A$	$8A$	$8B$	$8C$
$\Delta_{\mathrm{Ru}}(2A, nX, 29A)$	1914	11948	21489	5916	9860	17748
nX	$10A$	$10B$	$12A$	$12B$	$13A$	$14A$
$\Delta_{\mathrm{Ru}}(2A, nX, 29A)$	15196	30624	24128	48256	10904	21112
nX	$14B$	$14C$	$15A$	$16A$	$16B$	$20A$
$\Delta_{\mathrm{Ru}}(2A, nX, 29A)$	21112	21112	40223	37120	37120	28072
nX	$20B$	$20C$	$24A$	$24B$	$26A$	$26B$
$\Delta_{\mathrm{Ru}}(2A, nX, 29A)$	31320	31320	25288	25288	10904	10904
nX	$26C$	$29A$	$29B$	-	-	-
$\Delta_{\mathrm{Ru}}(2A, nX, 29A)$	10904	20735	20735	-	-	-

References

[1] F. Ali and M. A. F. Ibrahim, *On the ranks of HS and McL*, Utilitas Mathematica, to appear.

[2] F. Ali and M. A. F. Ibrahim , *On the ranks of Co_2 and Co_3*, submitted for publication.

[3] A. R. Ashrafi, *(p, q, r)-generations and nX-complementary generations of the Thompson group Th*, SUT J. Math. **39**(1) (2003), 41–54.

[4] A. R. Ashrafi and G. A. Moghani, *nX-complementary generations of the sporadic group Suz*, Bul. Acad. Ştiinţe Repub. Mold. Mat. **40**(3) (2002), 61–70.

[5] M. Aschbacher, *Sporadic Groups*, Cambridge Univ. Press, London-New York, 1994.

[6] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, and R. A. Wilson. *An Atlas of Finite Groups*, Oxford University Press, 1985.

[7] M. D. E. Conder, R. A. Wilson and A. J. Woldar, *The symmetric genus of sporadic groups*, Proc. Amer. Math. Soc. **116** (1992), 653–663.

[8] M. R. Darafsheh, and A. R. Ashrafi, *Generating pairs for the sporadic group Ru*, J. Appl. Math. & Computing, **10** (2002), 143–154.

[9] L. Di Martino and C. Tamburini, *2-Generation of finite simple groups and some related topics*, Generators and Relations in Groups and Geometry, A. Barlotti et al., Kluwer Acad. Publ., New York (1991), 195 – 233.

[10] J. I. Hall and L. H. Soicher, *Presentations of some 3-transposition groups*, Comm. Algebra **23** (1995), 2517-2559.

[11] M. A. F. Ibrahim and F. Ali, *On the ranks of* Co_1, submitted for publication.

[12] I. M. Isaacs, *Character Theory of Finite Groups*, Dover, New York, 1994.

[13] S. Linton, *The maximal subgroups of the sporadic groups* Th, Fi_{24} *and* Fi_{24}' *and other related topics*, Ph.D. thesis, Cambridge, 1989.

[14] J. Moori, *Generating sets for* F_{22} *and its automorphism group*, J. Algebra **159** (1993), 488–499.

[15] J. Moori, *Subgroups of* 3*-transposition groups generated by four* 3*-transpositions*, Quaest. Math. **17** (1994), 83–94.

[16] J. Moori, *On the ranks of the Fischer group* F_{22}, Math. Japonica , **43**(2) (1996), 365–367.

[17] J. Moori, *On the ranks of Janko groups* J_1, J_2 *and* J_3, Article presented at the 41st annual congress of Soth African Mathematical Society, RAU 1998.

[18] The GAP Group, $\mathbb{GAP}$ - *Groups, Algorithms and Programming, Version 4.3* , Aachen, St Andrews, 2003, (`http://www-gap.dcs.st-and.ac.uk/~gap`).

[19] L. L. Scott, *Matrices and cohomology*, Ann. Math. **105**(3) (1977), 473–492.

[20] A. J. Woldar, *Representing* M_{11}, M_{12}, M_{22} *and* M_{23} *on surfaces of least genus*, Comm. Algebra **18** (1990), 15–86.